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14 - Subset Lattices

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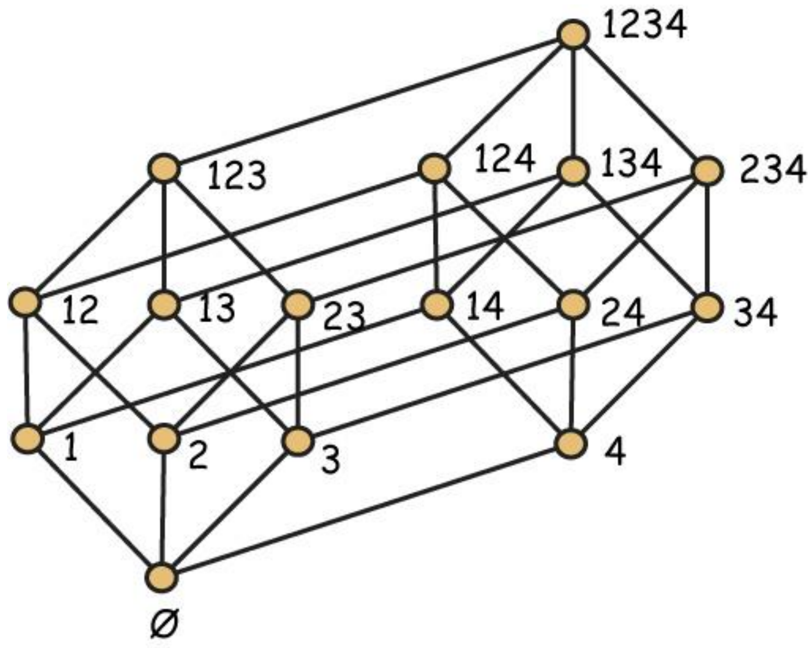
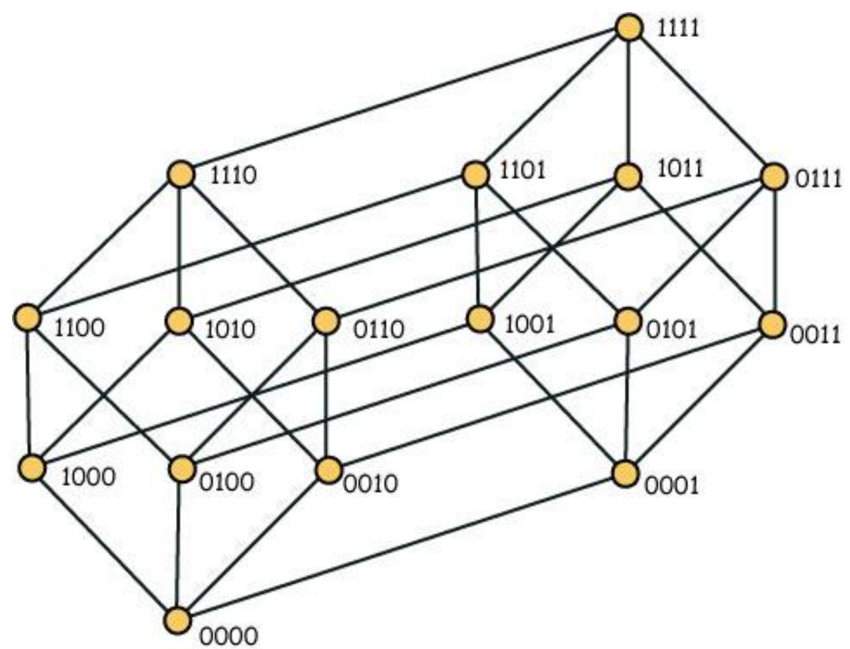
Subset Lattices

Definition For an integer $n \geq 1$, the poset consisting of all subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion is called the subset lattice. We will denote it as $\mathbf{2}^n$. Here is a diagram for $\mathbf{2}^4$.



Subset Lattices - Cubes

Remark Using the alternate notation for subsets as bit strings, subset lattices are also called cubes. Here is the 4-cube.



Basic Properties of Subset Lattices

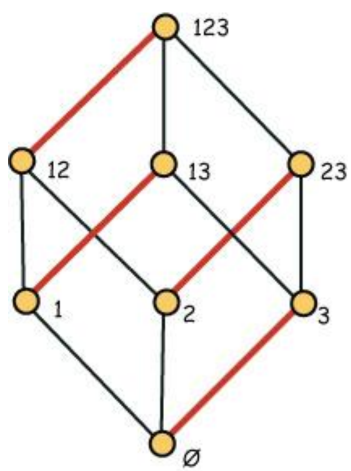
Fact The size of $\mathbf{2}^n$ is 2^n.

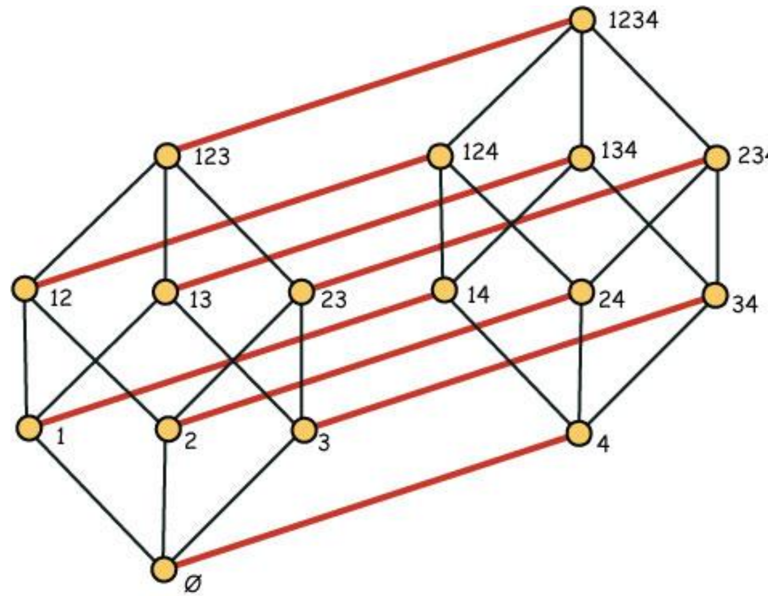
Fact The unique maximal element in $\mathbf{2}^n$ is the set $\{1, 2, \ldots, n\}$ and the unique minimal element is the empty set $\varnothing$.

Fact The height of $\mathbf{2}^n$ is $n + 1$. In fact, all maximal chains are maximum.

Inductive Nature of Subset Lattices

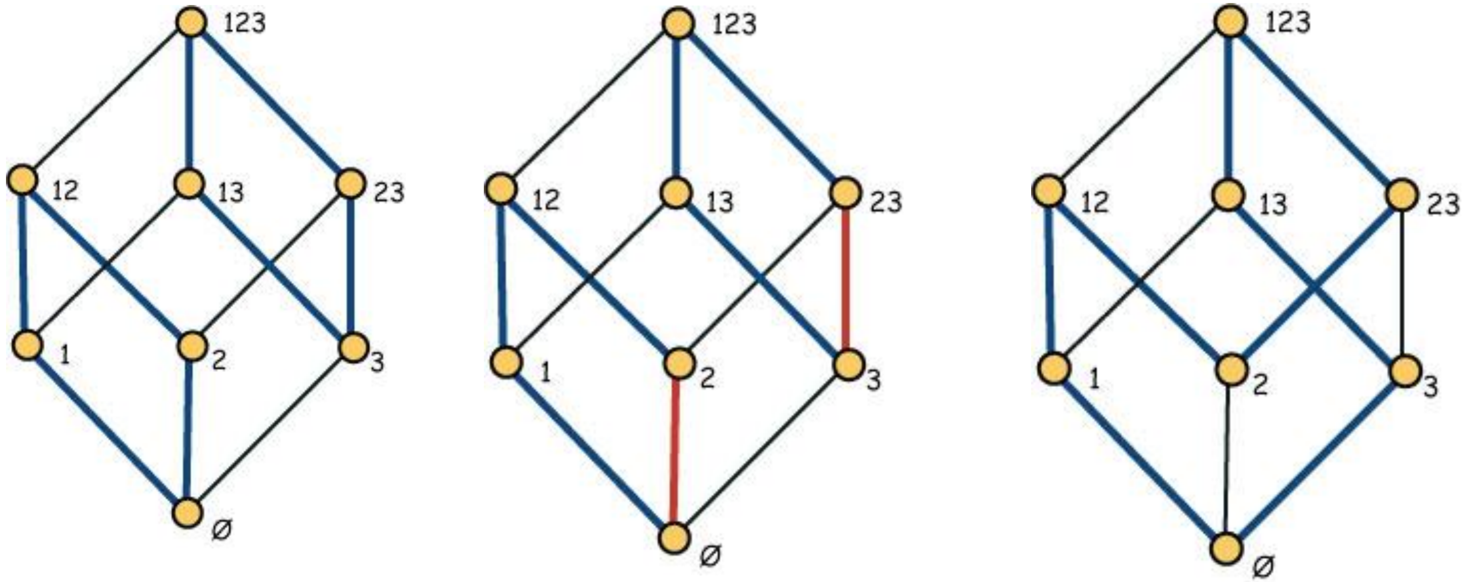
Basic Fact The subset lattice $\mathbf{2}^{n+1}$ can be viewed as $\mathbf{2} \times \mathbf{2}^{n}$.





Hamiltonian Property for Subset Lattices

Theorem For $n \geq 2$, the n-cube subset $\mathbf{2}^n$ is Hamiltonian.



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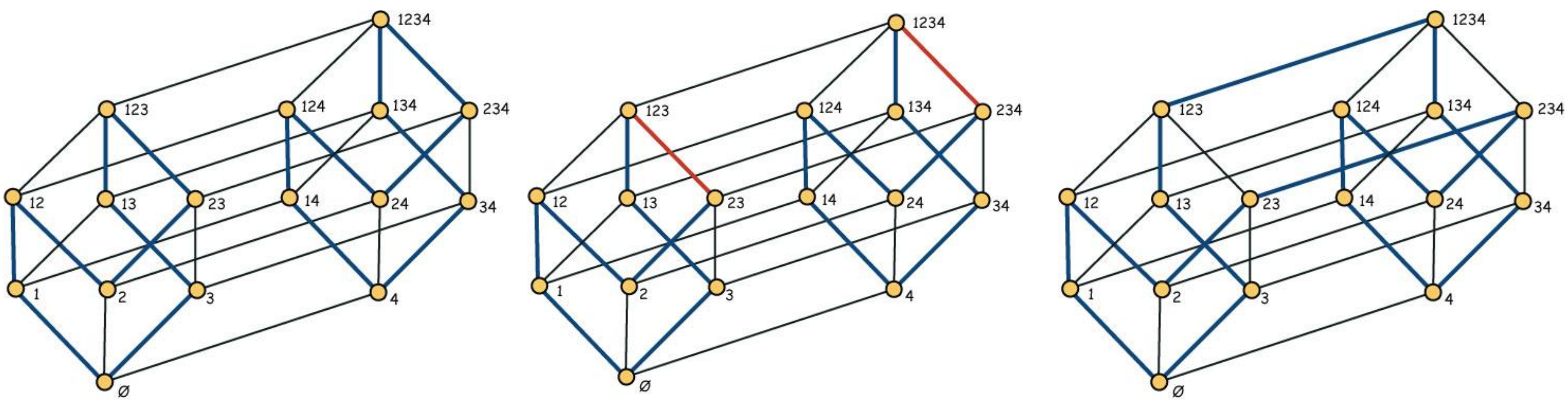


The Width of Subset Lattices

Fact If A is a set with $|A| = k$, then the number of maximal chains in $\mathbf{2}^n$ containing A is $k!\,(n-k)!$

Fact The width of $\mathbf{2}^n$ is at least as large as any binomial coefficient $C(n, k)$, where $0 \leq k \leq n$.

Fact The largest binomial coefficient of the form $C(n, k)$ is when $k = \lfloor n/2 \rfloor$. When n is even, there is just one value of k for which $C(n, k)$ is maximum. When n is odd, there are two. For example, the width of $\mathbf{2}^{13} \geq C(13, 6) = C(13, 7)$ while the width of $\mathbf{2}^{14} \geq C(14, 7)$.

The Width of Subset Lattices (2)

Theorem Fact (Sperner, '28) The width of the subset lattice $\mathbf{2}^n$ is the binomial coefficient $C(n, \lfloor n/2 \rfloor)$.

Note We will give two proofs of this result in class. The first proof is the more classical of the two and rests on the following elementary fact.

Fact If A is a subset of $\{1, 2, \ldots, n\}$ and $|A| = k$, then the number of maximal chains containing A is $k!\,(n-k)!$ To see this, consider bit strings. There are $k!$ ways to add the bits in A and then another $(n-k)!$ ways to add the bits in the complement of A.

The Width of Subset Lattices (3)

Proof of Spener's theorem Let $\{A_1, A_2, \ldots, A_t\}$ be a maximum antichain in $\mathbf{2}^n$. For each i, let $k_i = |A_i|$. Then

$$\sum_{1 \le i \le t} k_i!\,(n - k_i)! \le n!$$

$$\sum_{1 \le i \le t} [k_i!\,(n - k_i)!\,]/n! \le 1.$$

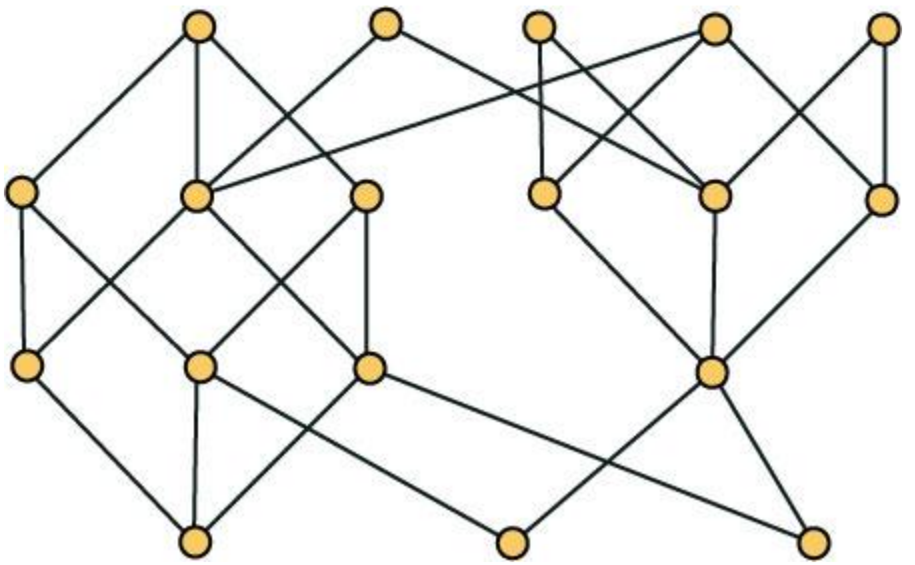
$$\sum_{1 \le i \le t} 1/C(n, k_i) \le 1$$

$$t\,/C(n, \lfloor n/2 \rfloor) \le 1$$

$$t \le C(n, \lfloor n/2 \rfloor)$$

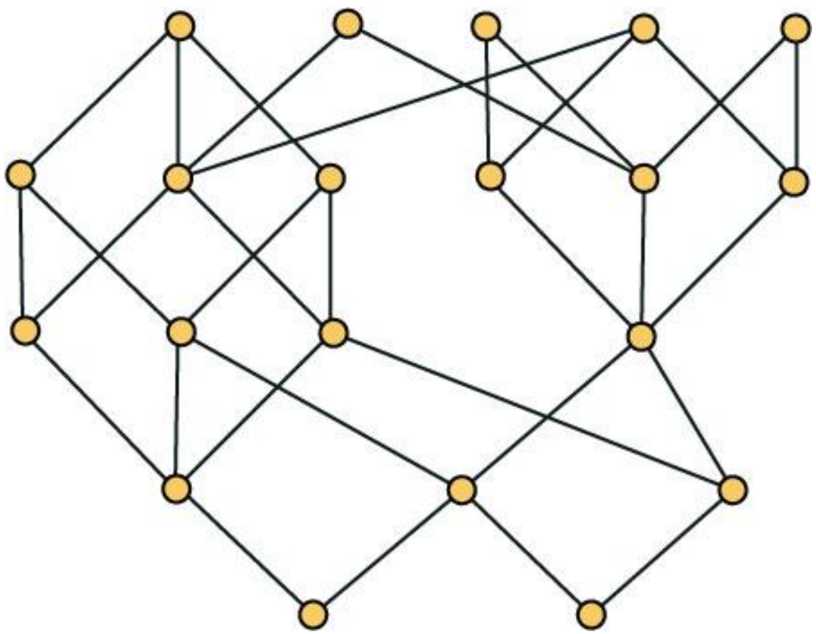
Ranked Posets – Start of Second Proof

Definition A poset is ranked if all maximal chains are maximum. Here is a ranked poset of height 4.



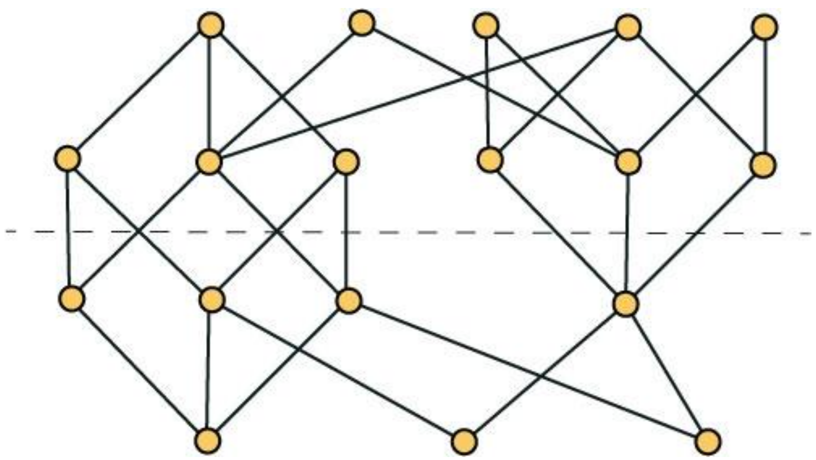
Ranked Posets (2)

Definition Here is a ranked poset of height 5.



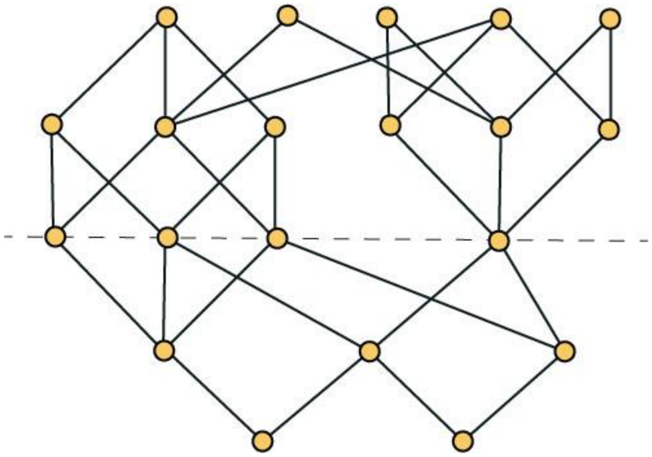
Middle Level of Ranked Poset

Observation Here is the middle level for a ranked poset of height 4.



Middle Level of Ranked Poset (2)

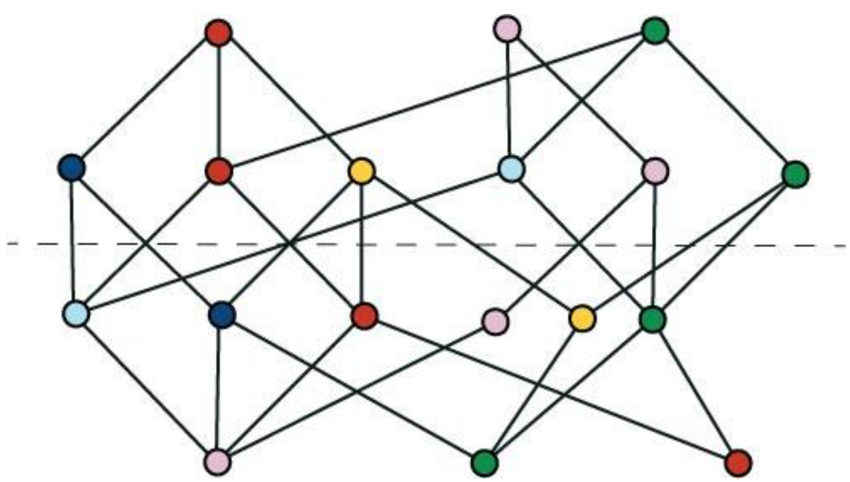
Observation Here is the middle level for a ranked poset of height 5.



Symmetric Chain Partition

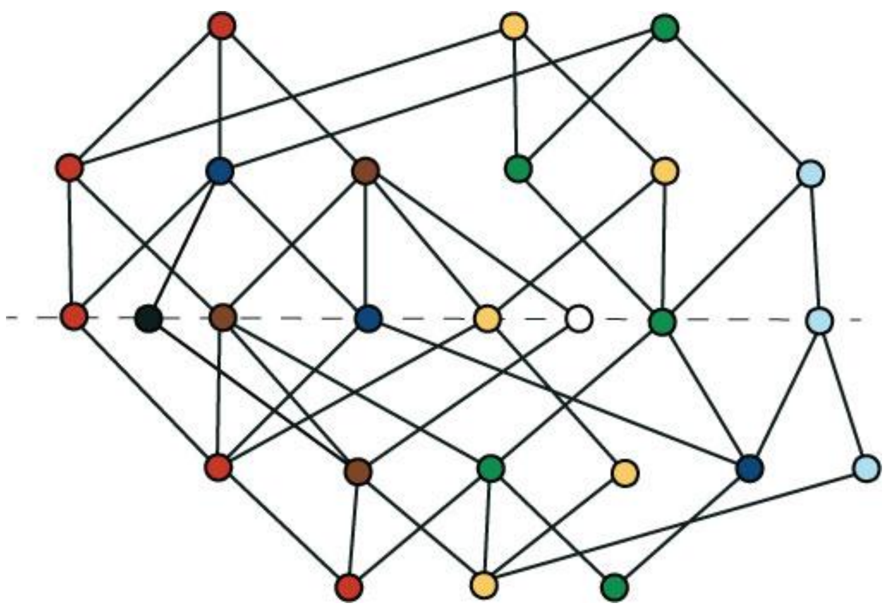
Definition A chain in a ranked poset is symmetric when it (1) goes the same distance above and below the middle levels and (2) doesn't skip levels.

Observation Here is a symmetric chain partition for a ranked poset of height 4.



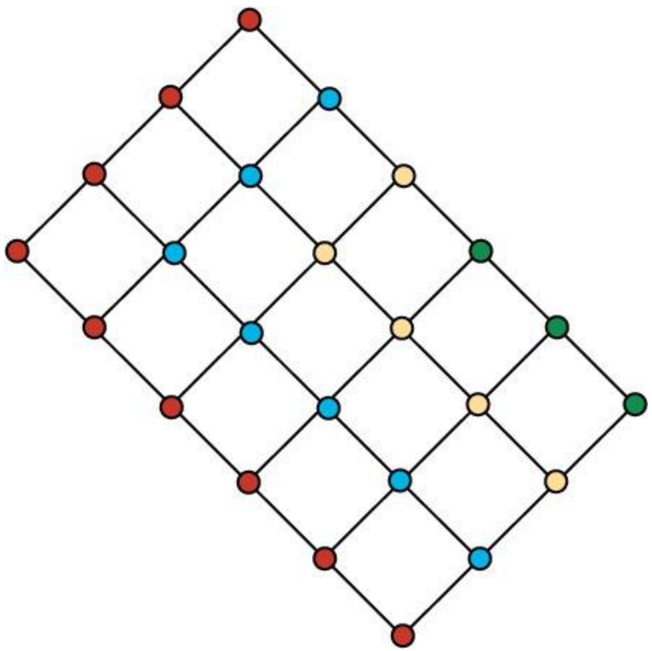
Symmetric Chain Partition (2)

Observation Here is a symmetric chain partition for a ranked poset of height 5.



Symmetric Chain Partition (3)

Lemma The Cartesian product of two chains has a symmetric chain partition.



Symmetric Chain Partition (4)

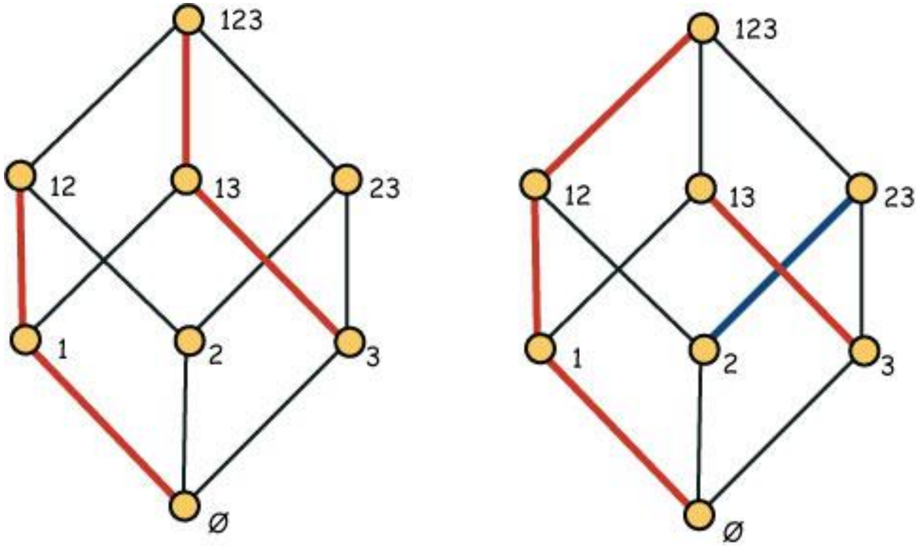
Theorem For every $n \geq 1$, the subset lattice $\mathbf{2}^n$ has a symmetric chain partition.

Proof On the next two slides, we illustrate the inductive construction. In each case, we start with a symmetric chain partition of $\mathbf{2}^n$ and show how to modify two copies to obtain a symmetric chain partition of $\mathbf{2}^{n+1}$.

Note that when n is even, we have some 1-element chains in the partition. Each pair becomes a 2-element chain in the next step. But when n is odd, each pair of chains produces another pair.

Symmetric Chain Partition (5)

Example Using two copies of a symmetric chain partition of $\mathbf{2}^2$ to form a symmetric chain partition of $\mathbf{2}^3$. Note that the 1-element chains 2 and 23 become a 2-element chain.



Symmetric Chain Partition (6)

Example Using two copies of a symmetric chain partition of $\mathbf{2}^3$ to form a symmetric chain partition of $\mathbf{2}^4$.