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23 - Applications of Probability to Combinatorics

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Foreword

Disclaimer Many of our examples will deal with games of chance and the notion of "gambling". This approach is historically accurate as a good fraction of early interest in probability has its roots in this setting. But this discussion is not intended in any way to be an endorsement of gambling. Instead, students should recognize that the underlying mathematical principles are highly relevant in real world problems in business, engineering, computer science, and just about any technical field of any significance. Very few things in life are absolutely certain, and it is very important to understand what is likely to occur and what is not.

Some Motivating Examples

Examples Which of the following games is fair?

Example 1 Alice rolls a pair of dice. If she rolls "doubles", then she scores 2 points. Otherwise, Dave scores $d - 2$ points where d is the difference.

Example 2 Bob flips a fair coin 10 times. He wins 3 points if he gets exactly five "heads" and five "tails". With any other outcome, Xing scores 1 point.

Example 3 Carlos rolls a single fair die. He wins if he rolls a "6". If he doesn't roll a "6", then he continues to roll as many times as it takes until he rolls a "6", and now Zori wins, or he rolls the same point as his first roll, in which case Carlos wins.

Some Motivating Examples (2)

Example 1 A new tv at Priceco costs $800. They offer you an insurance policy for $200 for full repair/replacement for the first three years. Is this a good option?

Example 2 Duck Fil A is considering 8 available locations on Northside drive and Howell Mill Road for a new franchise. Which one should they purchase.

Example 3 Next thanksgiving, how many turkeys should Trader Jack's purchase for their store at Monroe Drive?

Example 4 Is there any use whatsoever for a computer algorithm that returns the correct answer only 51% of the time?

Finite Probability Spaces

Definition A finite probability space Ω consists of a finite set X and a function P which assigns to each subset S of X a real number $P(S)$ such that:

1. $0 \le P(S) \le 1$ for every subset S of X.
2. $P(\emptyset) = 0$ and $P(X) = 1$.
3. If S and T are disjoint subsets of X, then $P(S \cup T) = P(S) + P(T)$.

Caution The symbol X will later be used in another way!

Finite Probability Spaces (2)

Observation If $X = \{x_1, x_2, \ldots, x_n\}$, then it is natural to abbreviate $P(\{x_i\})$ as $P(x_i)$. So to define the probability measure P, it is enough to know the values of $P(x_i)$ for each $i = 1, 2, \ldots, n$.

Example 1 $X = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$P(2) = P(12) = 1/36$

$P(3) = P(11) = 2/36$

$P(4) = P(10) = 3/36$

$P(5) = P(9) = 4/36$

$P(6) = P(8) = 5/36$

$P(7) = 6/36$

Finite Probability Spaces (3)

Example 2 Five cards are selected at random from a standard deck of 52 playing cards.

1. What is the probability that all five cards are hearts?
2. What is the probability of a “full house” (three of one kind and two of another)?
3. What is the probability of a “straight flush” (five cards in sequence all in the same suit)?

Finite Probability Spaces (4)

Example 3 An opaque jar contains 5 red marbles, 4 blue marbles and 7 green marbles. Two marbles are selected at random. What is the probability that both are red? Does it matter whether we draw both marbles at the same time … or draw one, record its color, put it back in the jar, shuffle and then draw again?

Example 4 Now consider a second opaque jar, identical on the outside. But this one contains 6 red marbles, 5 blue marbles and 4 green marbles. A jar is chosen at random and two marbles are drawn at random from that jar. What is the probability that both are red?

Bernoulli Trials

Bernoulli Trials An experiment either "succeeds" or it "fails". The probability of success is p and the probability of failure is $q = 1 - p$. The experiment is then repeated n times with different trials unaffected by the outcomes of the other trials.

Example 1 A fair coin is tossed with heads = success.

Example 2 A pair of die are rolled. Success is getting a even sum.

Example 3 Two cards are drawn from a deck. Success is that they are come from the same suit. The two cards must be placed back in the deck and the deck must be reshuffled before the next trial.

Bernoulli Trials (2)

Observation In a sequence of n Bernoulli trials with the probability of success being p, the probability that exactly m trials are successes is:

$$C(n, m)\, p^m (1 - p)^{n-m}$$

Example 1 A fair die is rolled 10 times. The probability of getting exactly a "3" on exactly two of the rolls is:

$$C(10, 2)\, (1/6)^2 (5/6)^8 = 1953125/6718464 \sim .29071$$

Example 2 The probability of five heads in 10 tosses of a fair coin is $C(10, 5)/2^{10} = 252/1024 \sim .24609\ldots$

Infinite Probability Spaces – Examples

Example In a Bernoulli trial set up, there are three outcomes x_1, x_2 and x_3 with probabilities p_1, p_2 and p_3, respectively. The trial is repeated until either x_1 occurs (this is a win) or x_2 occurs (this is a loss). As long as x_3 occurs, the trial is repeated. What is the probability of a win?

Answer

$p_1 + p_3 p_1 + p_3^2 p_1 + p_3^3 p_1 + p_3^4 p_1 + \ldots$

$= p_1(1 + p_3 + p_3^2 + p_3^3 + p_3^4 + \ldots.)$

$= p_1\ [1/(1 - p_3)]$

$= p_1/(p_1 + p_2)$

Infinite Probability Spaces – Examples (2)

Example 3 Carlos rolls a single fair die. He wins if he rolls a "6". If he doesn't roll a "6", then he continues to roll as many times as it takes until he rolls a "6", and now Zori wins, or he rolls the same point as his first roll, in which case Carlos wins. What is the probability that Carlos wins the game?

Answer

$1/6 + 5/6 \cdot 1/6 + 5/6 \cdot 4/6 \cdot 1/6 + 5/6 \cdot (4/6)^2 \cdot 1/6 + \ldots$

$= 1/6 + 5/6 \cdot 1/6\ (1 + 4/6 + (4/6)2 + (4/6)3 + \ldots)$

$= 1/6 + 5/6 \cdot 1/6\ [1/(1 - 4/6)]$

$= 1/6 + 5/6 \cdot 1/2$

$= 7/12 \sim .583333 \ldots$

Random Variables and Expectation

Definitions Given a finite probability space, a random variable is a function F which assigns to each element of the sample space a real number $F(x)$. The expected value of F, denoted $E(F)$, is the quantity:

$$F(x_1)p(x_1) + F(x_2)p(x_2) + F(x_3)p(x_3) + \ldots + F(x_n)p(x_n).$$

Example Let F assign the quantity m^2 when a single die is rolled and the number of spots showing is m. Then

$$E(F) = 1/6(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2)$$

$$= 91/6$$

$$= 15.33333 \ldots.$$

Caution Probabilists love to use symbols like X, Y, Z and W to denote random variables. Note the symbolic overload for X.

Random Variables and Expectation (2)

Example Suppose I have the opportunity to play the following game. First, I pay $15 each time I play the game. I then roll a single die and receive in return m^2 dollars where m is the number of spots on the die. Ethical issues aside, is this something I want to do?

Solution The expected value of my winnings each time I play is $15.33, against an investment of $15, so my expected net return is +33 cents. So I should play and play and play and play and play and play.

Caution If you think this way, you have to be prepared for substantial deviations in the expected behavior.

Returning to Our Motivating Examples

Example 1 Alice rolls a pair of dice. If she rolls "doubles", then she scores 2 points. Otherwise, Dave scores d – 2 points where d is the difference.

Solution Let X be the expected value of Alice's score. Then

$E(X) = 1/6 \cdot 2 + 10/36 \cdot 1 + 8/36 \cdot 0 - 6/36 \cdot 1 - 4/36 \cdot 2 - 2/36 \cdot 3$

$= 2/36$

$= 1/18$

Alice is in the winning position!

Returning to Our Motivating Examples (2)

Example 2 Bob flips a fair coin 10 times. He wins 3 points if he gets exactly five "heads" and five "tails". With any other outcome, Xing scores 1 point.

Solution Let X be the expected value of Bob's score. The probability of getting exactly five heads and five tails in a total of 10 flips of a fair coin is $C(10, 5)/2^{10} \sim .24609\ldots$

So $E(X) = 3\,(.24609) - 1(\,1 - .24609)$

$= .73827 - 75391$

$= -.01564$

So Bob is in the losing position!

Returning to our Motivating Examples (3)

Example 3 Carlos rolls a single fair die. He wins if he rolls a "6". If he doesn't roll a "6", then he continues to roll as many times as it takes until he rolls a "6", and now Zori wins, or he rolls the same point as his first roll, in which case Carlos wins.

Solution Previously we calculated the probability that Carlos wins as 7/12. Since this is more than 1/2, the game is biased in favor of Carlos.

One Final Comment

Remark The expected return on a $1 investment in a Las Vegas casino is slightly less than 90 cents, i.e., long term you should expect to lose slightly more than 10% of what you risk. People gamble while knowing these irrefutable mathematical facts just for the thrill that comes with occasional "jackpot".

Remark The expected return from state sponsored lotteries is considerably less than what one would expect in a casino. This is especially true for the "power ball" big ticket lotteries where the expected return on a $1 investment is close to 50 cents. Of course, people play the Georgia Lottery because they support the ways the state promises to spend the profits they make.