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Math 3012 – Applied Combinatorics Lecture 2

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The Road Ahead

Alert The next two to three lectures will be an integrated approach to material from Chapters 2, 3 and 4. Please read these three Chapters – in order – concurrently or even in advance of the discussions we will have in class. Homework assignments will be posted, with odd ones to enhance the pace of your understanding as full solutions will be available. *A* few even number problems will be assigned and these will be collected for grading.

An Introduction to Strings

Let n be a positive integer and let $[n] = \{1, 2, \ldots, n\}$. A sequence of length n such as $(a_1, a_2, \ldots, a_n)$ is called a **string** (also a **word**, an **array** or a **vector**).

The entries in a string are called **characters**, **letters**, **coordinates**, etc. The set of possible entries is called the **alphabet**.

Examples

010010100010110011101 - a bit string

201002211001020 – a ternary string

abcacbaccbbaaccbabaddbbadcabbd – a word from a four letter alphabet.

NHZ 4235 – *A* Georgia auto license plate

I love mathematics (really)!! - a word from an alphabet with 59 letters – upper and lower cases, spaces and punctuation.

Arrays in Computer Languages

Example

```
int A[10];
for (i = 0; i < 10; i++) {
  A[i] = 2*i + 3;
}
```

Display 35791113151719 21 (Bad formatting!)

3, 5, 7, 9, 11, 13, 15, 17, 19, 21 (Better)

3 5 7 9 11 13 15 17 19 21 (Even better)

The First Principle of Enumeration

Observation If a project can be considered as a sequence of n tasks which are carried out in order, and for each i, the number of ways to do Task i is m_i, then the total number of ways the project can be done is:

$$m_1 \times m_2 \times m_3 \times \dots m_n$$

Consequences

Fact The number of bit strings of length n is 2^n.

Fact The number of words of length n from an m letter alphabet is m^n.

Fact The number of Georgia license auto license plates is $26^3 10^4$.

Permutations – Repetition not allowed

Examples

12 7 8 6 4 9 11	Yes
X y a A D 7 B E 9	Yes
5 b 7 2 4 9 A 7 6 X	No

Fact The number of permutations of length n from an m letter alphabet is: $P(m, n) = m\,(m-1)\,(m-2)\,\ldots\,(m-n+1)$.

Language $P(m, n)$ is the number of permutations of m objects taken n at a time.

How to Answer a Question

Question How many permutations of 68 objects taken 23 at a time?

Answer P(68, 23)

Comment In almost all situations, I want you to stop right there and leave it to the dedicated reader to determine exactly what the value of P(68, 23) turns out to be. After all, this is just arithmetic. However, if you're really curious, P(68, 23) turns out to be:

20732231223375515741894286164203929600000

Permutations and Combinations

Contrasting Problems

Problem 1 A group of 250 students holds elections to identify a class president, a vice-president, and a treasurer. How many different outcomes are possible.

Problem 2 A group of 250 students holds elections to select a leadership committee consisting of three persons. How many different outcomes are possible?

Permutations and Combinations

Solutions

Problem 1 A group of 250 students holds elections to identify a class president, a vice-president, and a treasurer. How many different outcomes are possible.

Answer $P(250, 3) = 250 * 249 * 248$

Permutations and Combinations

Solutions

Problem 2 A group of 250 students holds elections to select a leadership committee consisting of three persons. How many different outcomes are possible?

Answer $C(250, 3) = \dfrac{250 * 249 * 248}{1 * 2 * 3}$

Read this answer as the number of combinations of 250 objects, taken 3 at a time.

Binomial Coefficients

In Line Notation

C(38, 17) = P(38, 17)/17! = 38!/(21! 17!)

Graphic Notation

$$\binom{\mathbf{38}}{\mathbf{17}}$$

Read this: "38 choose 17"

Binomial Coefficients

Basic Definition

$$\binom{38}{17} = \frac{38\,!}{17\,!21!}$$

Note To compute this binomial coefficient, you have to do a lot of multiplication and some division. Maybe there is an alternative way??!!

Beware Dot, dot, dot!!!

Question What is the next term: 1, 4, 9, 16, 25 ?

Question What is the next term: 1, 1, 2, 3, 5, 8, 13?

Question What is the sum 1 + 2 + 3 + … + 6 ?

Question What is really meant by the definitions:

$$n! = n * (n - 1) * (n - 2) * \dots 3 * 2 * 1$$

$$P(m, n) = m * (m - 1) * (m - 2) * \dots * (m - n + 1)$$

A Better Way

Observation Rather than writing $1, 4, 9, 16, 25, \ldots$

be explicit and write: $a_n = n^2$

Observation Rather than writing $1, 1, 2, 3, 5, 8, 13, \ldots$

be explicit and write:

$a_1 = 1$; $a_2 = 1$;
and when $n \geq 3$, $a_n = a_{n-2} + a_{n-1}$.

A Better Way

Observation Rather than writing $1 + 2 + \ldots 6$, say "the sum of the first six positive integer."

Observation An even better way:
Define $S_0 = 0$ and when $n \geq 1$, $S_n = n + S_{n-1}$.
Then reference S_6.

A Better Way

Definition $0! = 1$ and when $n > 1$,

$n! = n * (n-1)!$

Example

$5! = 5 * 4!$

$4! = 4 * 3!$

$3! = 3 * 2!$

$2! = 2 * 1!$

$1! = 1 * 0!$

Backtracking We obtain

$1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$ and $5! = 120$

A Better Way

Definition $P(m, 1) = m$ and when $1 < n \leq m$,
$P(m, n) = (m - n + 1) * P(m, n - 1)$.

Example
$P(7, 4) = (7 - 4 + 1) * P(7, 3) = 4 * P(7, 3)$
$P(7, 3) = (7 - 3 + 1) * P(7, 2) = 5 * P(7, 2)$
$P(7, 2) = (7 - 2 + 1) * P(7, 1) = 6 * P(7, 1)$
$P(7, 1) = 7$

Backtracking We obtain
$P(7, 2) = 6 * 7 = 42$
$P(7, 3) = 5 * 42 = 210$
$P(7, 4) = 4 * 210 = 840$

Coding Basics

Declaration

```
int  factorial ( int  n);
```

Definition

```
int  factorial { int  n) {
   if (n == 0) return 1;
   else return  (n) * factorial (n - 1);
}
```

Coding Basics

Declaration

```
int  permutation (int  m, int  n);
```

Definition

```
int  permutation {int  m, int  n) {
   if (n == 1) return m;
   else return  (m – n + 1) * permutation(m, n – 1);
}
```

Bit-strings and Subsets

Equivalent Problems

Problem 1 How many bit strings of length 38 have exactly 17 ones?

Problem 2 How many subsets of size 17 in a set of size 38?

Answer

$$C(38, 17) = P(38, 17)/17! = 38!/(21!\ 17!)$$

Basic Identities

Complements

$$\binom{n}{k} = \binom{n}{n-k}$$

Eliminating Multiplication

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

Pascal's Triangle

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Combinatorial Identities

First Grade Formula

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$$

Second Grade Formula

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

Third Grade Formula

$$\binom{n}{0} 2^n + \binom{n}{1} 2^{n-1} + \binom{n}{2} 2^{n-2} + \dots + \binom{n}{n} 2^0 = 3^n$$

Enumerating Distributions

Basic Enumeration Problem

Given a set of m objects and n cells (boxes, bins, etc.), how many ways can they be distributed?

Side Constraints

1. Distinct/non-distinct objects
2. Distinct/non-distinct cells
3. Empty cells allowed/not allowed.
4. Upper and lower bounds on number of objects in a cell.

Binomial Coefficients Everywhere

Foundational Enumeration Problem

Given a set of m identical objects and n distinct cells, the number of ways they can be distributed with the requirement that no cell is empty is

$$\binom{\boldsymbol{m-1}}{\boldsymbol{n-1}}$$

Explanation

A A A A A A | A A | A A A A | A A A A A A A | A | A A A

m objects, m – 1 gaps. Choose n – 1 of them. In this example, there are 23 objects and 6 cells. We have illustrated the distribution (6, 2, 4, 7, 1, 3).