Student Name and ID Number

MATH 3012 Final Exam, May 3, 2013, WTT

1. Consider the 62-element set X consisting of the twenty-six letters (case sensitive) of the English alphabet and the ten digits $\{0, 1, 2, \ldots, 9\}$.

a. How many strings of length 15 can be formed if repetition of symbols is permitted?

b. How many strings of length 15 can be formed if repetition of symbols is *not* permitted?

c. How many strings of length 15 can be formed using exactly four A's, two a's, seven 3's and two 5's?

d. How many strings of length 15 can be formed if exactly seven characters are capital letters, exactly four characters are 6's and the remaining four characters are digits? Here, repetition is allowed.

e. How many binary relations are there on X?

f. How many equivalence relations are there on X with class sizes 15, 15, 6, 6, 6, 6, 6 and 2?

2. How many integer valued solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 62$ when:

a. $x_i > 0$ for $i = 1, 2, 3, 4, 5$.

b. $x_i \geq 0$ for $i = 1, 2, 3, 4, 5$.

c. $x_i > 0$ for $i = 1, 2, 3, 5$ and $x_4 > 9$.

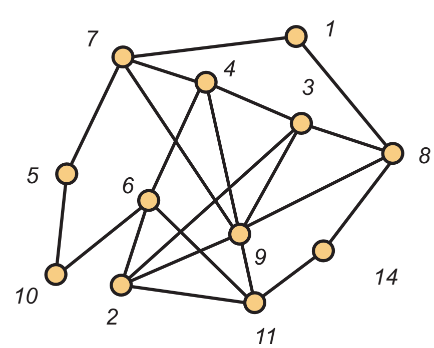
d. $x_i > 0$ for $i = 1, 2, 3, 4, 5$ and $x_4 \leq 9$.

3 a. Use the Euclidean algorithm to find $d = \gcd(252, 1320)$.

b. Use your work in the first part of this problem to find integers a and b so that $d = 252a + 1320b$.

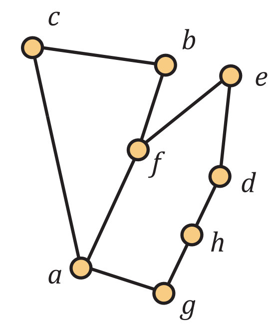
4. For a positive integer n, let s_n count the number of ternary sequences which do not have three consecutive 2's. Note that $s_1 = 3$, $s_2 = 9$, $s_3 = 26$ and $s_4 = 76$. For $n \geq 5$, develop a recurrence for s_n and use it to find s_6.

5. Use the algorithm developed in class to find an Euler circuit in the graph G shown below (use node 1 as root):



6. Consider again the graph from the preceding problem. List the vertices in an order that shows why the graph is also hamiltonian.

7. Consider the graph G shown below.

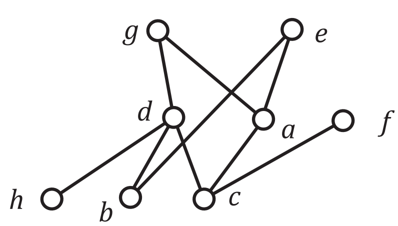


a. Determine $\omega(G)$.

b. Show that $\chi(G) = \omega(G)$ by providing a proper coloring of G. You may indicate your coloring by writing directly on the figure.

c. Explain why the graph G is perfect.

8. Consider the poset P shown below.



a. Draw a circle around the ordered pairs in the following list that belong to the reflexive, antisymmetric and transitive binary relation which defines this poset.

(d, b) (h, e) (a, e) (f, f) (c, g) (e, b) (b, b)

b. The poset P is not an interval order. By inspection, find four points which determine a subposet isomorphic to $\mathbf{2} + \mathbf{2}$.

c. What is the width of the poset P?

d. List a set of elements which forms a maximum antichain in P.

e. Find a Dilworth partition of the poset P. You may provide your answer by writing directly on the figure.

9. For the subset lattice $\mathbf{2}^{17}$,

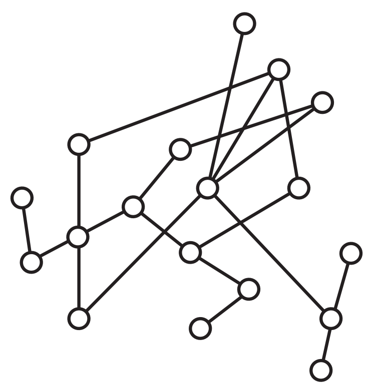
a. The total number of elements is:

b. The total number of maximal chains is:

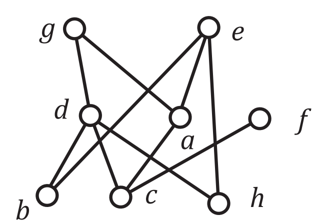
c. The number of maximal chains through $\{3, 5, 8, 11\}$ is:

d. The width of $\mathbf{2}^{17}$ is:

10. For the poset P shown below, find the height h and a partition into h antichains by recursively stripping off the set of maximal elements. You may display your answer by writing directly on the diagram. Then darken a set of points that form a maximum chain.



11. The poset P shown below is an interval order:



a. Find the down sets and the up sets. Then use these answers to find an interval representation of P that uses the least number of end points.

$D(a) =$	$U(a) =$	$I(a) =$
$D(b) =$	$U(b) =$	$I(b) =$
$D(c) =$	$U(c) =$	$I(c) =$
$D(d) =$	$U(d) =$	$I(d) =$
$D(e) =$	$U(e) =$	$I(e) =$
$D(f) =$	$U(f) =$	$I(f) =$
$D(g) =$	$U(g) =$	$I(g) =$
$D(h) =$	$U(h) =$	$I(h) =$

b. In the space below, draw the representation you have found. Then use the First Fit Coloring Algorithm for interval graphs to solve the Dilworth Problem for this poset, i.e., find the width w and a partition of P into w chains. You may display your answers by writing the colors directly on the intervals in the diagram.

c. Find a maximum antichain in P:

12 a. Write all the partitions of the integer 7 into odd parts:

b. Write all the partitions of the integer 7 into distinct parts:

13a. Find the general solution to the advancement operator equation:
$(A-2)^3(A+5)^2(A-3)(A+7)f = 0$

b. Write the form of a particular solution of the non-homogeneous advancement operator equation (do not carry out the work necessary to evaluate any constants in your answer):
$(A-2)^3(A+5)^2(A-3)(A+7)f = 4(3)^n$

c. Find the solution to the advancement operator equation:
$(A^2 - 5A + 6)f(n) = 0, \quad f(0) = 3$ and $f(1) = 14$.

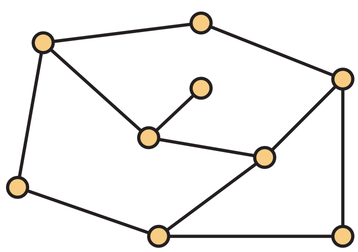
14a. Write the inclusion/exclusion formula for the number $S(n, m)$ of onto functions from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, m\}$.

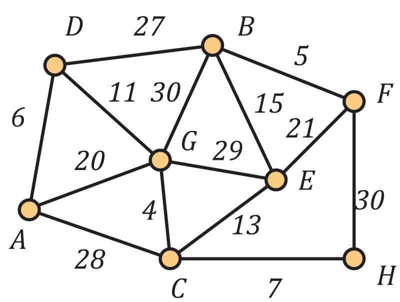
b. Write the inclusion/exclusion formula for the number d_n of derangements on $\{1, 2, \ldots, n\}$.

c. Evaluate your formula for d_n when $n = 6$.

d. Find the value of the Euler ϕ-function $\phi(n)$ when $n = 2^3 \cdot 5 \cdot 7^2$.

15a. Verify Euler's formula for the planar graph shown on the left below.





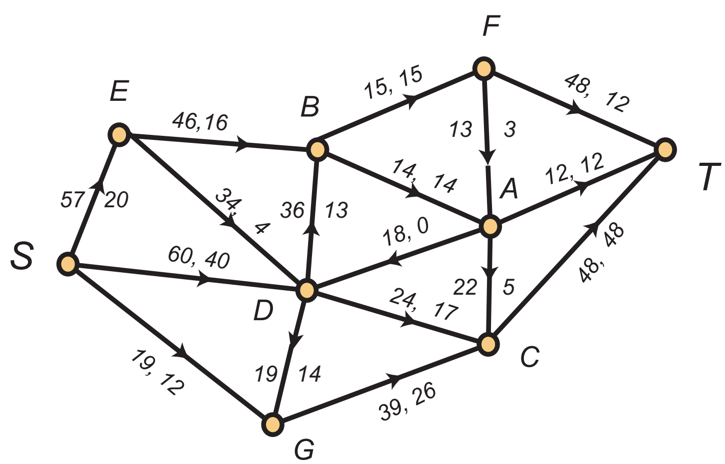
b. Now consider the weighted graph shown on the right above. In the space below, list *in order* the edges which make up a minimum weight spanning tree using, respectively Kruskal's Algorithm (avoid cycles) and Prim's Algorithm (build tree). For Prim, use vertex A as the root.

Kruskal's Algorithm

Prim's Algorithm

16. Consider again the weighted graph from the preceding problem. Consider the weights as lengths, with all edges capable of being traversed in either direction. Apply Dijkstra's algorithm to find the distance from vertex A to all other vertices in the graph. Also, for each vertex X, find a shortest path from A to X.

17. Consider the following network flow:



a. What is the current value of the flow?

b. What is the capacity of the cut $V = \{S, B, D, E, G\} \cup \{A, C, F, T\}$.

c. Carry out the labeling algorithm, using the pseudo-alphabetic order on the vertices and list below the labels which will be given to the vertices.

d. Use your work in part c to find an augmenting path and make the appropriate changes directly on the diagram.

e. Carry out the labeling algorithm a second time on the updated flow. It should halt without the sink being labeled.

f. Find a cut whose capacity is equal to the value of the updated flow.

18. Consider a poset P whose ground set is $X = \{a, b, c, d, e, f, g, h, i\}$. Network flows (and the special case of bipartite matchings) are used to find the width w of P and a minimum chain partition. When the labelling algorithm halts, the following edges are matched:

$$e'h'' \quad c'f'' \quad d'e'' \quad h'b''$$

a. Find the chain partition of P that is associated with this matching.

b. Find the width w of the poset P.

c. Explain why elements a, i and g belong to every maximum antichain in P.

19. True–False. Mark in the left margin.

1. $2^{40} > 100,000,000$
2. There is a planar graph G on 328 vertices with $\chi(G) = 9$.
3. All graphs with 986 vertices and 4073 edges are non-planar.
4. There is a non-hamiltonian graph on 684 vertices in which every vertex has degree 426.
5. Every connected graph on 783 vertices in which every vertex has degree 12 has an Euler circuit.
6. A cycle on 548 vertices is a homeomorph of the complete bipartite graph $K_{2,2}$.
7. When $n \geq 3$, the shift graph S_n has $\binom{n}{2}$ vertices, and $\binom{n}{3}$ edges. Furthermore, $\chi(S_n) = \lceil \lg n \rceil$.
8. The number of lattice paths from $(0,0)$ to (n,n) which do not pass through a point above the diagonal is the Catalan number $\binom{2n}{n}/(n+1)$.
9. Any modern computer can accept a file of $3,000$ positive integers, each at most $5,000$, and quickly determine whether $3,742$ is the sum of two integers in the file.
10. Any modern computer can accept a file of $3,000$ positive integers, each at most $5,000$, and quickly determine whether $385,742$ is the product of two integers in the file.
11. Any modern computer can accept a file of $3,000$ positive integers, each at most $5,000$, and quickly factor each of the numbers into primes.
12. There is a graph on 782 vertices in which no two vertices have the same degree.
13. There is a poset with 723 points having width 69 and height 9.
14. There is a sequence of 923 distinct positive integers which does not have an increasing subsequence of size 21 nor a decreasing subsequence of size 41.
15. The permutation $(7,1,3,5,2,4,6)$ is a derangement.
16. The number of equivalence relations on a set of size 100 is less than $10,000$.
17. The binary relation $R = \{(a,a),(a,b),(b,b),(c,c)\}$ is reflexive on $X = \{a,b,c\}$.
18. The binary relation $R = \{(a,a),(a,b),(c,c),(b,c)\}$ is transitive on $X = \{a,b,c\}$.
19. The binary relation $R = \{(a,a),(a,b),(c,c),(b,a)\}$ is antisymmetric on $X = \{a,b,c\}$.
20. The binary relation $R = \{(a,a),(b,b),(c,c),(a,b),(b,a)\}$ is an equivalence relation on $X = \{a,b,c\}$.
21. Linear programming problems with integer coefficient constraints always have integer valued solutions.
22. Every linear programming problem is also a network flow problem.