Solutions

Student Name and ID Number

MATH 3012 Final Exam, December 12, 2017, WTT

9

3x3

1. Consider the 16-element set X consisting of the ten digits $\{0, 1, 2, \ldots, 9\}$ and the capital letters $\{A, B, C, D, E, F\}$.

a. How many strings of length 12 can be formed if repetition of symbols is *not* permitted?

$P(16, 12)$

b. How many strings of length 12 can be formed if repetition of symbols is permitted?

16^{12}

c. How many strings of length 12 can be formed using exactly four 2's, three 5's, and five B's?

$\binom{12}{4,3,5}$ OR $\frac{12!}{4!\,3!\,5!}$

9

3x3

2. How many integer valued solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 48$ when:

a. $x_i > 0$ for $i = 1, 2, 3, 4, 5$.

$\binom{47}{4}$

b. $x_1, x_2, x_5 > 0$, $x_3, x_4 \geq 0$.

$\binom{49}{4}$

c. $x_1, x_2, x_3, x_5 > 0$, $0 < x_4 \leq 5$.

$\binom{47}{4} - \binom{42}{4}$

10

2x5

3 a. Use the Euclidean algorithm to find $d = \gcd(2592, 828)$.

$828 \overline{)2592}$ quotient 3: $2592 - 2484 = 108$

$108 \overline{)828}$ quotient 7: $828 - 756 = 72$

$72 \overline{)108}$ quotient 1: $108 - 72 = 36$

$36 \overline{)72}$ quotient 2: $72 - 72 = 0$

$\gcd = 36$

b. Use your work in the first part of this problem to find integers a and b so that $d = 2592a + 828b$.

$2592 = 3 \cdot 828 + 108$ $\qquad$ $108 = 1 \cdot 2592 - 3 \cdot 828$

$828 = 7 \cdot 108 + 72$ $\qquad$ $72 = 1 \cdot 828 - 7 \cdot 108$

$108 = 1 \cdot 72 + 36$ $\qquad$ $36 = 1 \cdot 108 - 1 \cdot 72$

$36 = 1 \cdot 108 - 1(1 \cdot 828 - 7 \cdot 108)$

$= 8 \cdot 108 - 1 \cdot 828$

$= 8(1 \cdot 2592 - 3 \cdot 828) - 1 \cdot 828$

$= 8 \cdot 2592 - 25 \cdot 828$

$\boxed{a = 8 \qquad b = -25}$

8

4x2

4. For the subset lattice $\mathbf{2}^{16}$,

a. The total number of elements is: 2^{16}

b. The total number of maximal chains is: $16!$

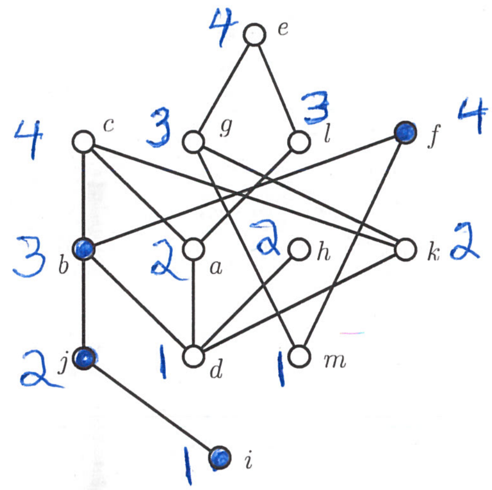
c. The number of maximal chains containing $\{2, 5, 6, 7, 13\}$ is: $5!\,11!$

d. The width of $\mathbf{2}^{16}$ is: $\binom{16}{8}$

12

2,2,2,4,2

5. For the poset P shown below,



a. List all elements comparable with b. c, f, j, d, i

b. List all elements covered by b. j, d

c. By inspection (not by algorithm), explain why this poset is not an interval order.
It contains $\mathbf{2}+\mathbf{2}$ e.g. $a > d$ and $g > m$. (Many others)

d. Find the height h and a partition into h minimal elements by recursively stripping off the set of minimal elements. Please indicate your partition by coloring directly on the diagram using the integers in $[h]$ so that for each $i \in [h]$, all elements colored i form an antichain. The height of P is: 4

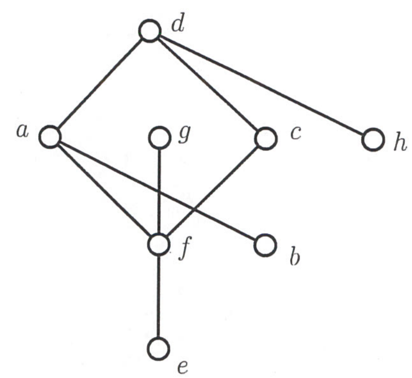
e. On the diagram, darken a set of h points that form a maximum chain.

There are at least two other correct ways

12

~~4,6~~ 5,5,2

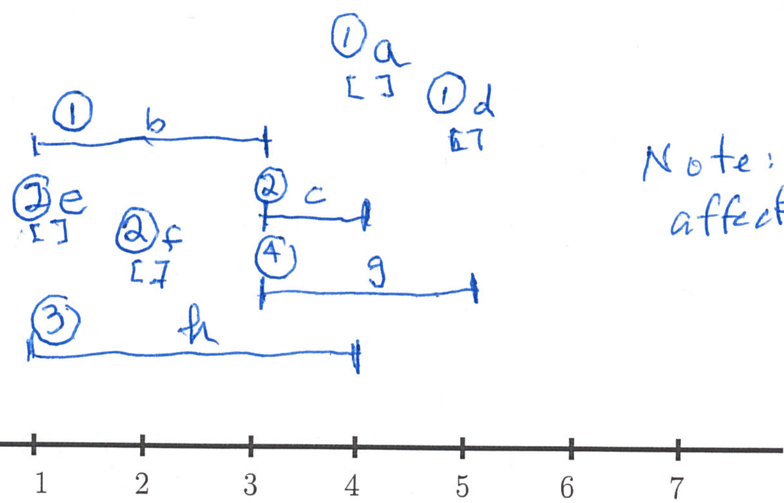
6. The poset P shown below is an interval order:



a. Find the down sets and the up sets. Then use these answers to find an interval representation of P that uses the least number of end points.

$D(a) =$ b f e	$U(a) =$ d	$I(a) =$ [4 , 4]
$D(b) =$ ∅	$U(b) =$ a d	$I(b) =$ [1 , 3]
$D(c) =$ f e	$U(c) =$ d	$I(c) =$ [3 , 4]
$D(d) =$ a b c f e h	$U(d) =$ ∅	$I(d) =$ [5 , 5]
$D(e) =$ ∅	$U(e) =$ a c d f g	$I(e) =$ [1 , 1]
$D(f) =$ e	$U(f) =$ a c d g	$I(f) =$ [2 , 2]
$D(g) =$ f e	$U(g) =$ ∅	$I(g) =$ [3 , 5]
$D(h) =$ ∅	$U(h) =$ d	$I(h) =$ [1 , 4]

b. In the space below, draw the representation you have found. Then use the First Fit Coloring Algorithm for interval graphs to solve the Dilworth Problem for this poset, i.e., find the width w and a partition of P into w chains. Please display your answers by writing the colors directly on the intervals in your diagram.



Note: Coloring affected by drawing.

c. Find a maximum antichain in P:

~~b, c, h~~ b, c, g, h

width = 4

10

7. For a positive integer n, let t_n count the number of ternary sequences which do not have 20 as a substring of two consecutive terms. Note that $t_1 = 3$ and $t_2 = 8$. Develop a recurrence for t_n and use it to find t_5.

$\boxed{\quad\quad|0}$ t_{n-1} − bad. Bad = $\boxed{\text{good } |2|0}$ $= t_{n-2}$

$\boxed{\quad\quad|1}$ t_{n-1}

$\boxed{\quad\quad|2}$ t_{n-1}

Recurrence: $t_n = 3t_{n-1} - t_{n-2}$

$t_3 = 3\cdot 8 - 3 = 24 - 3 = 21$

$t_4 = 3\cdot 21 - 8 = 63 - 8 = 55$

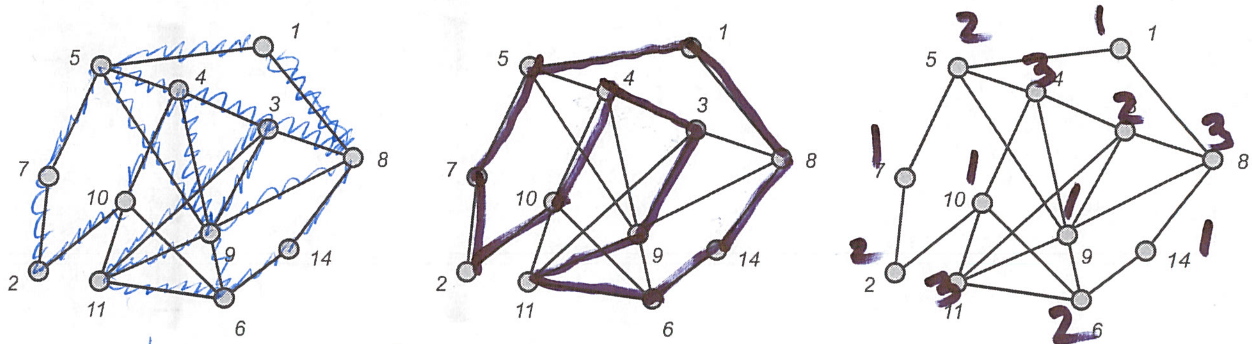
$t_5 = 3\cdot 55 - 21 = 165 - 21 = 144$

12

6, 3, 3

8. Three copies of a graph G are given below.

a. Use the algorithm presented in class (always proceed to the least neighbor using an edge not yet visited) to find an Euler circuit in the graph G shown below (use node 1 as root). If you mark any edges in your work, please use the copy on the left.



(1, 5, 4, 3, 8, 1)

5, 7, 2, 10, 4, 9, 3, 11, 6, 9, 5

(1, 5, 7, 2, 10, 4, 9, 3, 11, 6, 9, 5, 4, 3, 8, 1)

(10, 6, 14, 8, 9, 11, 10)

(1, 5, 7, 2, 10, 6, 14, 8, 9, 11, 10, 4, 9, 3, 11, 6, 9, 5, 4, 3, 8, 1)

b. List the vertices of G in an order that shows why G is also hamiltonian. Alternatively, you may darken an appropriate set of edges on the second copy.

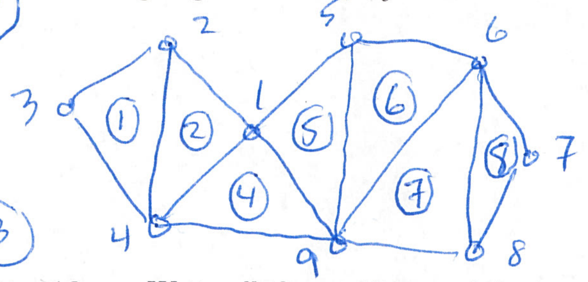
Other correct solutions

c. On the third copy, give a three coloring by writing directly on this copy with colors from [3].

Essentially everything is forced except colors on vertices 2 and 7.

8

9. Draw a connected planar graph with 9 vertices and 15 edges in a manner that there are no crossing edges. Then verify Euler's formula for your drawing.



3

$V - E + F = 2$

$9 - 15 + 8 = 2$

$17 - 5 = 2$ ✓

12

3×4

10 a. Write all the partitions of the integer 8 into odd parts:

$8 = 7+1$
$= 5+3$
$= 5+1+1+1$
$= 3+3+1+1$
$= 3+1+1+1+1+1$
$= 1+1+1+1+1+1+1+1$

6 in total

same

b. Write all the partitions of the integer 8 into distinct parts:

$8 = 8$
$= 7+1$
$= 6+2$
$= 5+3$
$= 5+2+1$
$= 4+3+1$

6 in total

c. Write in product form the generating function for the number of partitions of the integer n into even parts with no part being repeated more than twice.

$f(x) = (1+x^2+x^4)(1+x^4+x^8)(1+x^6+x^{12})(1+x^8+x^{16})(1+x^{10}+x^{20})\cdots$

8

11. Find the general solution to the advancement operator equation:
$(2-3i)(A+7-i)^3(A-5)^2 f = 0$

root $-7+i$ multiplicity 3, 5 mult 2

$f(n) = c_1(-7+i)^n + c_2 n(-7+i)^n + c_3 n^2(-7+i)^n + c_4 5^n + c_5 n 5^n$

10

12. Find the solution to the advancement operator equation:
$(A^2+3A-10)f(n) = 0, \quad f(0) = 11$ and $f(1) = 1$.

$(A+5)(A-2)$

root $-5, 2$

$f(n) = c_1(-5)^n + c_2 2^n$

$f(0) = c_1 + c_2 = 11$

$f(1) = -5c_1 + 2c_2 = 1$

$-2c_1 - 2c_2 = -22$

$-7c_1 = -21$

$c_1 = 3 \quad c_2 = 8$

$$\boxed{f(n) = 3(-5)^n + 8 \cdot 2^n}$$

12

6+2

13 a. How many permutations of the integers in [13] satisfy $\sigma(2) = 2$, $\sigma(5) = 5$ and $\sigma(9) = 9$?

Fix 3 $\qquad 10!$

b. How many functions $f : [10] \to [6]$ omit 2 and 5 from the range?

4 possibilities $\qquad 4^{10}$

c. If $n = 3^4 \cdot 11^2 \cdot 19^5$, how many integers in $[n]$ are divisible by 57? Note that $57 = 3 \cdot 19$.

$\frac{n}{57}$ or $\frac{n}{3 \cdot 19}$

d. Write the inclusion/exclusion formula for the number of onto functions (surjections) from $[n]$ to $[m]$.

$$S(n,m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n$$

e. Write the inclusion/exclusion formula for the number d_n of derangements of $[n]$.

$$d_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$$

f. Write the inclusion/exclusion formula for the number $\phi(n)$ of integeers in $[n]$ which are relatively prime to n when $n = 7^4 \cdot 11^3 \cdot 13^5$.

$$\phi(n) = n\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1-\frac{1}{13}\right)$$

12

2+6

14. Consider the data file (shown on the left below) for the weights on the edges of a graph with vertex set $\{a, b, c, d, e, f, g, h\}$. In the space to the right, list in order the edges that would be selected in carrying out Kruskal's algorithm (avoid cycles) and Prim's algorithm (build tree) to find a minimum weight spanning tree. For Prim, use vertex a as the root.

graphdata.txt

e	h	12
c	d	14
b	c	15
c	e	19
c	h	23
d	h	25
a	b	27
d	e	29
a	g	32
f	g	35
b	h	39

Kruskal

eh
cd
bc
ce
ab
ag
fg

Prim

ab
bc
cd
ce
eh
ag
fg

12

15. A data file `digraph_data.txt` has been read for a digraph whose vertex set is $\{1, 2, 3, 4, 5, 6, 7\}$. The weights on the directed edges are shown in the matrix below. The entry $w(i, j)$ denotes the length of the edge from i to j. If there is no entry, then the edge is not present in the graph. In the space to the right, apply Dijkstra's algorithm to find the distance from vertex 1 to all other vertices in the graph. Also, for each x, find a shortest path from 1 to x.

W	1	2	3	4	5	6	7
1	0		10		45	9	
2		0		20	22		50
3		7	0	35	30		60
4		20		0	2		10
5					0		
6		10		38	36	0	
7							0

2
1,2 w=∞

$\boxed{\text{3} \quad \text{1,3 w=10}}$

4
1,4 w=∞

5
1,5 w=45

$\boxed{\text{6 P} \quad \text{1,6 w=9}}$

7
1,7 w=∞

1,6,2
w=9+10=19

$\boxed{\text{1,3,2} \quad \text{w=10+7=17}}$

1,6,4
w=9+38
=47

~~1,3,5~~

$\boxed{\text{1,3,2,4} \quad \text{w=17+20=37}}$

$\boxed{\text{1,3,2,5} \quad \text{w=17+2} \quad \text{=39}}$

1,3,7
w=10+60
=70

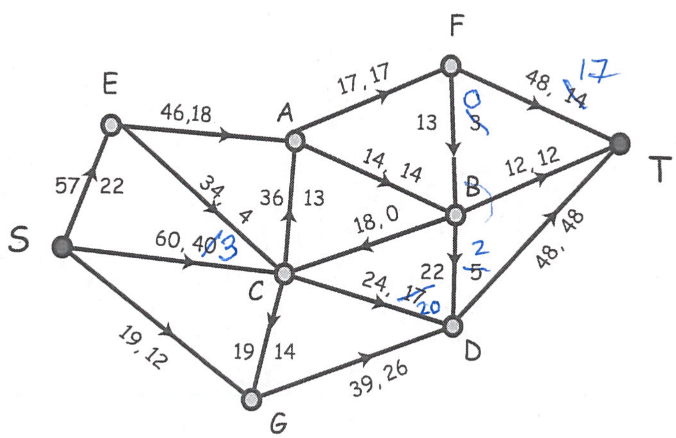
1,3,2,7
w=17+50
=67

$\boxed{\text{1,3,2,4,7} \quad \text{w=37+10} \quad \text{=47}}$

12

6×2

16. Consider the following network flow:



a. What is the current value of the flow?

22 + 40 + 12 = 74

b. What is the capacity of the cut $V = \{S, B, D, E, G\} \cup \{A, C, F, T\}$.

46 + 34 + 60 + 18 + 48 + 12

c. Carry out the labeling algorithm, using the pseudo-alphabetic order on the vertices and list below the labels which will be given to the vertices.

S(*, +, ∞) A(C, +, 20) T(F, +, 3)
C(S, +, 20) D(C, +, 7)
E(S, +, 35) B(D, −, 5)
G(S, +, 7) F(B, −, 3)

d. Use your work in part c to find an augmenting path and make the appropriate changes directly on the diagram.

S, C, D, B, F, T

e. Carry out the labeling algorithm a second time on the updated flow. It should halt without the sink being labeled.

S(*, +, ∞) A(C, +, 17)
C(S, +, 17) D(C, +, 4)
E(S, +, 35) B(D, −, 2)
G(S, +, 7)

f. Find a cut whose capacity is equal to the value of the updated flow.

S, C, E, G, A, D, B | F, T

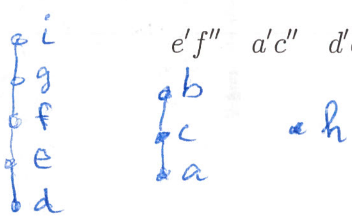
10

2+5

17. Consider a poset P whose ground set is $X = \{a, b, c, d, e, f, g, h, i\}$. Network flows (and the special case of bipartite matchings) are used to find the width w of P and a minimum chain partition.

a. What chain partition of P is associated with the following matching (which is not maximum):

$e'f''$ $a'c''$ $d'e''$ $f'g''$ $c'b''$ $g'i''$

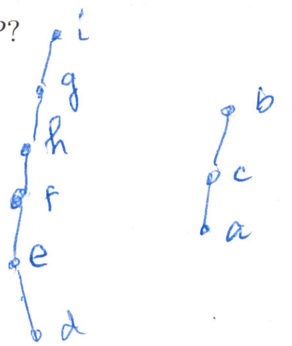


b. When the labelling algorithm halts, the following edges are matched:

$c''b''$

$e'f''$ $a'c''$ $d'e''$ $f'h''$ $c'h''$ $g'i''$ $h'g''$

What is the width of P?



width = 2

18. An opaque jar contains 3 red marbles, 4 blue marbles and 2 green marbles. After being shaken, two marbles are drawn (at the same time), and the payout in dollars is the number of blue marbles. What is the expected value of the payout.

6

$P(0) = \frac{\binom{5}{2}}{\binom{9}{2}} = \frac{10}{36}$ $\qquad E = \frac{10}{36} \cdot 0 + \frac{20}{36} \cdot 1 + \frac{6}{36} \cdot 2$

$P(1) = \frac{4 \cdot 5}{36} = \frac{20}{36}$ $\qquad = \frac{32}{36}$

$P(2) = \frac{\binom{4}{2}}{\binom{9}{2}} = \frac{6}{36}$ $\qquad = \frac{8}{9}$

16

16×1

19. True–False. Mark in the left margin.

T 1. $2^{100} > 100,000,000,000,000,000,000,000$.

F 2. There is a planar graph G on 619 vertices with $\omega(G) = 3$ and $\chi(G) = 12$.

T 3. All graphs with 985 vertices and 4723 edges are non-planar.

F 4. There is a non-hamiltonian graph on 482 vertices in which every vertex has degree 246.

T 5. Every connected graph on 482 vertices in which every vertex has degree 246 has an Euler circuit.

F 6. The number of lattice paths from $(7, 10)$ to $(18, 14)$ is $\binom{25}{24}$.

T 7. A cycle on 48 vertices is a homeomorph of the complete bipartite graph $K_{2,2}$.

T 8. When $n \geq 2$, the shift graph S_n has $\binom{n}{2}$ vertices and $\binom{n}{3}$ edges.

F 9. The chromatic number of the shift graph S_n is $\binom{n}{\lfloor n/2 \rfloor}$.

T 10. The number of lattice paths from $(0,0)$ to (n,n) which do not pass through a point above the diagonal is the Catalan number $\binom{2n}{n}/(n+1)$.

F 11. There is a poset with 823 points having width 78 and height 9.

F 12. There is a sequence of 243 distinct positive integers which does not have an increasing subsequence of size 5 nor a decreasing subsequence of size 61.

F 13. The permutation $(8, 4, 1, 7, 5, 2, 6, 3)$ is a derangement.

F 14. Linear programming problems with integer coefficient constraints always have integer valued solutions.

T 15. Every network flow problem is also a linear programming problem.

F 16. Every linear programming problem is also a network flow problem.

TOTAL 200 Points