KEY

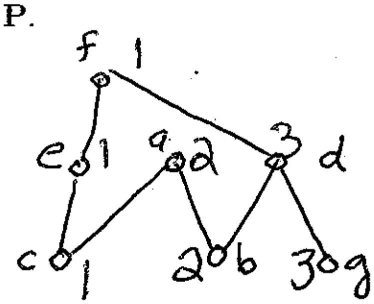
Student Name and ID Number

MATH 3012 Quiz 2, March 8, 2007, WTT

1. Define a poset $\mathbf{P} = (X, P)$ with $X = \{a, b, c, d, e, f, g\}$ by setting $P = \{(x, x) : x \in X\} \cup Q$, with

$$Q = \{(e, f), (d, f), (g, d), (b, d), (b, a), (c, e), (c, a), (c, f), (g, f), (b, f)\}$$

4 **a.** Draw a poset diagram for $\mathbf{P}$.



4 **b.** Find the width w of P. Also, find a set of w points from P that form an antichain.

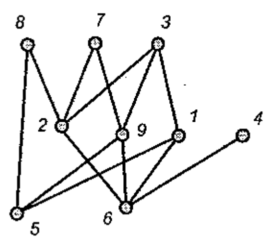
$w = 3$ $\{a, d, e\}$, $\{e, b, g\}$, $\{c, b, g\}$, $\{a, e, g\}$

4 **c.** Find a partition into w antichains by marking the points in the diagram (in part a) with integers in $\{1, 2, \ldots, w\}$ so that all points marked with the same integer form a chain. as marked

4 **d.** Show that $\mathbf{P}$ is not an interval order by finding four points in $\mathbf{P}$ that induce a copy of $\mathbf{2} + \mathbf{2}$.

$\{e, c\}$ $\{d, g\}$, ALSO $\{a, c\}$, $\{d, g\}$

2. Consider the following poset.



a. This poset is an interval order. For each $i = 1, 2, \ldots, 9$, find the down set $D(i)$ and the upset $U(i)$.

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3 $D(1) = \{5, 6\}$
2 $D(2) = \{6\}$
6 $D(3) = \{1, 2, 5, 6, 9\}$
2 $D(4) = \{6\}$
1 $D(5) = \emptyset$
1 $D(6) = \emptyset$
5 $D(7) = \{2, 5, 6, 9\}$
4 $D(8) = \{2, 5, 6\}$
3 $D(9) = \{5, 6\}$

5 $U(1) = \{3\}$
3 $U(2) = \{3, 7, 8\}$
4 $U(3) = \emptyset$
6 $U(4) = \emptyset$
2 $U(5) = \{1, 3, 7, 8, 9\}$
1 $U(6) = \{1, 2, 3, 4, 7, 8, 9\}$
6 $U(7) = \emptyset$
6 $U(8) = \emptyset$
4 $U(9) = \{3, 7\}$

4 **b.** How many distinct down sets does this poset have? 6

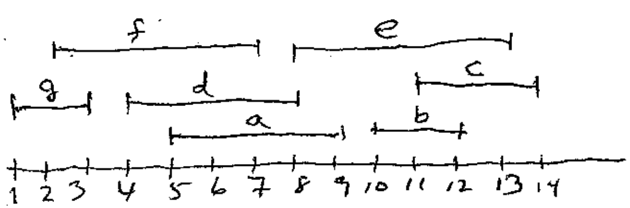
4 **c.** How many distinct up sets does this poset have? 6

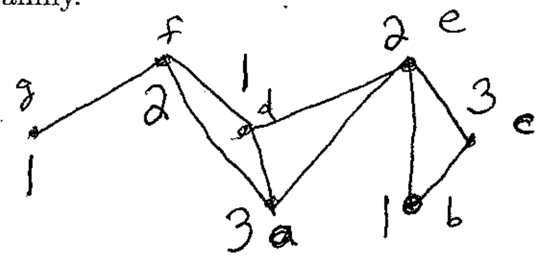
(8) **d.** If m is the number of distinct down sets, find the unique interval representation of P using intervals with integer endpoints from $\{1, 2, \ldots, m\}$. Then use the First Fit algorithm to color the interval graph associated with this set of intervals (process the vertices in the order of their left end points).

$I(1) =$ [3, 5] $I(2) =$ [2, 3]
$I(3) =$ [6, 6] $I(4) =$ [2, 6]
$I(5) =$ [1, 2] $I(6) =$ [1, 1]
$I(7) =$ [5, 6] $I(8) =$ [4, 6]
$I(9) =$ [3, 4]

3. Consider the family of intervals: $I(a) = [5, 9]$, $I(b) = [10, 12]$, $I(c) = [11, 14]$, $I(d) = [4, 8]$, $I(e) = [8, 13]$, $I(f) = [2, 7]$ and $I(g) = [1, 3]$.

(4) **a.** Draw the interval graph G_1 determined by this family.



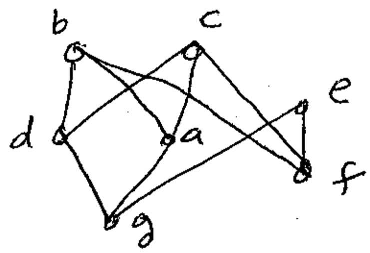


(4) **b.** Use the First Fit Algorithm to color the vertices in this graph, processing the vertices in the order of their left end points. Record the results directly on your drawing of the graph.

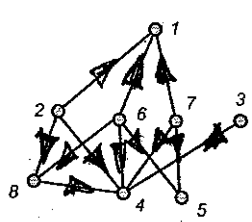
(4) **c.** Let k be the number of colors used by First Fit in the preceding step. We know that $k \geq \chi(G_1) \geq \omega(G_1)$. Show that $k = \chi(G_1) = \omega(G_1)$ by finding a clique of size k in G_1.

$\{a, d, f\}$ $\{a, d, e\}$ $\{b, c, e\}$

(4) **d.** Draw a diagram for the interval order determined by this family.

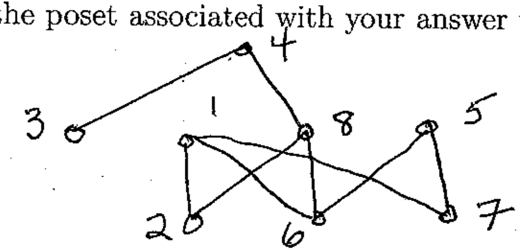


4. Consider the graph G_2 shown below.

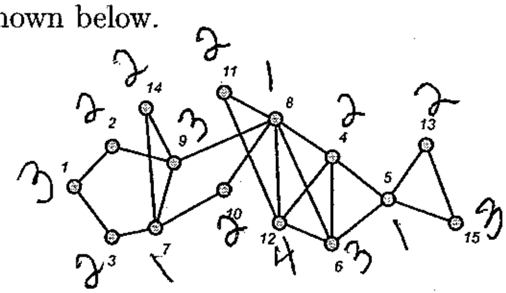


(4) **a.** Show that G_2 is a comparability graph by orienting the edges transitively.

(4) **b.** Draw the diagram of the poset associated with your answer to part a.



5. Consider the graph G_3 shown below.



(4) a. Find the maximum clique size $\omega(G_3)$ of this graph. Also find a set of $\omega(G_3)$ vertices that forms a clique.
$\omega(G_3) = 4$ $\{4, 6, 8, 12\}$

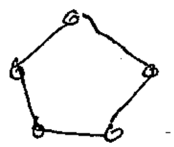
(4) b. Color the vertices of G_3 with $\omega(G_3)$ colors so that adjacent vertices never receive the same color. Note: this shows that $\chi(G_3) = \omega(G_3)$.

(4) c. What is the length of the largest cycle contained in G_3? Find a set of vertices that forms a cycle of this length. length = 7 $\{1, 2, 9, 8, 10, 7, 3\}$

(4) d. What is the length of the largest *induced* cycle contained in G_3? Find a set of vertices that forms an induced cycle of this length. length = 5 $\{1, 2, 9, 7, 3\}$

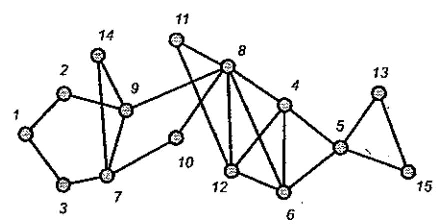
(4) e. Find a sequence of vertices forming a shortest path from vertex 2 to vertex 6.

(4) f. We have already noted that $\chi(G_3) = \omega(G_3)$. Nevertheless, G_3 is not perfect. Explain why.



C_5 is an induced subgraph of G_3. But $\chi(C_5) = 3 > \omega(C_5) = 2$

6. We again consider the graph G_3 used in the preceding problem:



(8) a. Show that G_3 has an eulerian circuit by applying the "stage" algorithm developed in our class.

Stage 1 1, 2, 9, 7, 3, 1

Stage 2 1, 2, 9, 8, 4, 5, 6, 4, 12, 8, 10, 7, 14, 9, 7, 3, 1

Stage 3 1, 2, 9, 8, 11, 12, 6, 8, 4, 5, 6, 4, 12, 8, 10, 7, 14, 9, 7, 3, 1

Stage 4 1, 2, 9, 8, 11, 12, 6, 8, 4, 5, 13, 15, 5, 6, 4, 12, 8, 10, 7, 14, 9, 7, 3, 1

(4) b. This graph does not have a hamiltonian cycle, but it does have a hamiltonian path, i.e., a path that includes each vertex exactly once, starting at vertex 13 and ending at vertex 14. List the vertices in an order that forms such a hamiltonian path.

$13, 15, 5, \underbrace{4, 6}_{\text{or } 6, 4}, 12, 11, 8, 10, 7, 3, 1, 2, 9, 14$