KEY

Student Name and ID Number

MATH 3012 Section F, Quiz 3, April 9, 2009, WTT

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1. Find the general solution to the advancement operator equation:
$(A+2)^3(A-5)^2(A+4)f(n) = 0$

$f(n) = c_1(-2)^n + c_2 n(-2)^n + c_3 n^2(-2)^n + c_4 5^n + c_5 n 5^n + c_6(-4)^n$

2. Find the solution to the advancement operator equation:
$(A^2+2A-15)f(n) = 0, \quad f(0) = 1$ and $f(1) = -61$.

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$A^2 + 2A - 15 = (A+5)(A-3)$

$f(n) = c_1(-5)^n + c_2 3^n$

$c_1 + c_2 = 1$

$-5c_1 + 3c_2 = -61$

$8c_2 = -56$

$c_2 = -7$

$c_1 = 8$

$f(n) = 8(-5)^n - 7\cdot 3^n$

3. Find a particular solution to the advancement operator equation:
$(A-4)f(n) = 5\cdot 3^n$.

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$f(n) = c\cdot 3^n$

$c\cdot 3^{n+1} - 4c\cdot 3^n = 5\cdot 3^n$

$-c\,3^n = 5\cdot 3^n$

$c = -5$

$f(n) = -5\cdot 3^n$

4. Write the inclusion/exclusion formula for the number of onto functions from $\{1,2,\ldots,m\}$ to $\{1,2,\ldots,n\}$. Then calculate the answer when $m = 8$ and $n = 3$.

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$\sum_{k=0}^{n} (-1)^k \binom{n}{k}(n-k)^m = 3^8 - 3\cdot 2^8 + 3\cdot 1$

$\binom{3}{0}3^8 - \binom{3}{1}2^8 + \binom{3}{2}1^8$

5. Write the inclusion/exclusion formula for the number of derangements on $\{1,2,\ldots,n\}$. Then evaluate the formula when $n = 7$.

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$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)!$

$\binom{7}{0}7! - \binom{7}{1}6! + \binom{7}{2}5! - \binom{7}{3}4! + \binom{7}{4}3! - \binom{7}{5}2! + \binom{7}{6}1! - \binom{7}{7}0!$

$= 21\cdot 120 - 35\cdot 24 + 35\cdot 6 - 21\cdot 2 + 7\cdot 1 - 1\cdot 1$

6. Let n be an integer with $n \geq 2$ and let $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ be the factorization of n into primes. Write the inclusion/exclusion formula for the Euler function $\phi(n)$. Then use this formula to find $\phi(90)$.

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$\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_k}\right)$

$90 = 2\cdot 3^2\cdot 5$

$\phi(90) = 90\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right) = \cancel{90}^{\cancel{30}\,6}\cdot\frac{1}{\cancel{2}}\cdot\frac{2}{\cancel{3}}\cdot\frac{4}{\cancel{5}} = 24$

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7. Use generating functions to prove that for every $n \geq 1$, the number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.

$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\ldots$

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$= \frac{\cancel{1-x^2}}{1-x} \cdot \frac{\cancel{1-x^4}}{\cancel{1-x^2}} \cdot \frac{\cancel{1-x^6}}{1-x^3} \cdot \frac{\cancel{1-x^8}}{\cancel{1-x^4}} \cdot \frac{\cancel{1-x^{10}}}{1-x^5}$ — — — —

$= \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^7} \ldots$

$= (1+x+x^2+x^3+\ldots)(1+x^3+x^6+x^9+\ldots)(1+x^5+x^{10}+x^{15}+\ldots)(1+x^7+x^{14}+\ldots)\ldots$

First function is generating function for # of partitions into distinct parts

last " " " " " " " " odd parts

8. a. Calculate $\binom{\frac{1}{2}}{3}$ $\quad \frac{\frac{1}{2}\cdot(\frac{1}{2}-1)(\frac{1}{2}-2)}{1\cdot 2\cdot 3} = \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{\cancel{6}\,2} = \frac{1}{16}$

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b. Write the general form of the binomial theorem:
$(1+x)^p = \sum_{n=0} \binom{p}{n} x^n$

c. Let $(1-4x)^{-1/2} = \sum_{n=0}^{n=\infty} a_n x^n$. What is the coefficient a_n in this generating fucntion: $a_n = \binom{2n}{n}$

d. Write the identity that results from squaring the series from part c and equating coefficients with the geometric series for $1/(1-4x)$:

$$4^n = \sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k}$$

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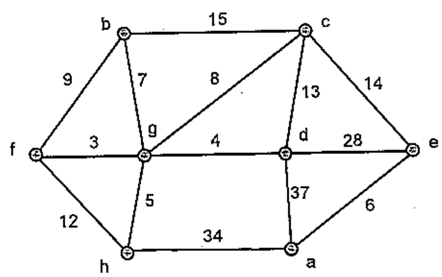

In the space below, list *in order* the edges which make up a minimum weight spanning tree using, respectively Kruskal's Algorithm (avoid cycles) and Prim's Algorithm (build tree). For Prim, use vertex a as the root.

Kruskal's Algorithm

f g 3
g d 4
g h 5
a e 6
g b 7
g c 8
c e 14

Prim's Algorithm

a e 6
c e 14
c g 8
f g 3
g d 4
g h 5
g b 7

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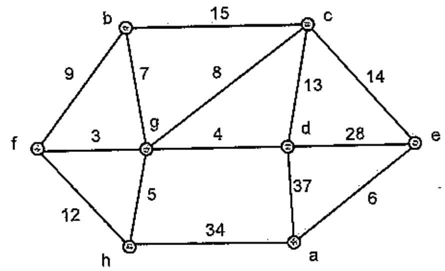

For the graph shown above, traffic is allowed to flow in either direction on the edges. Carry out Dijkstra's algorithm to find shortest paths from node a to all other nodes.

	b	c	d	e	f	g	h
	∞ (a,b)	∞ (a,c)	37 (a,d)	$\boxed{6\ (a,e)}$	∞ (a,f)	∞ (a,g)	34 (a,h)
scan e	∞ (a,b)	$\boxed{20\ (a,e,c)}$	34 (a,e,d)		∞ (a,f)	∞ (a,g)	34 (a,h)
Scan c	35 (a,e,c,b)		33 (a,e,c,d)		∞ (a,f)	$\boxed{28\ (a,e,c,g)}$	34 (a,h)
Scan g	35 (a,e,c,b)		32 (a,e,c,g,d)		$\boxed{31\ (a,e,c,g,f)}$		33 a,e,c,g,h
Scan f	35 (a,e,c,b)		$\boxed{32\ (a,e,c,g,d)}$				33 (a,e,c,g,h)
Scan d	35 (a,e,c,b)						$\boxed{33\ a,e,c,g,h}$
Scan h	$\boxed{35\ (a,e,c,b)}$						

TOTAL 56 + 30 + 14 = 100