Solutions

Student Name and ID Number

MATH 3012 Quiz 1, February 8, 2013, WTT

(12) 4×3

1. Consider the 62-element alphabet consisting of the ten digits $\{0, 1, 2, \ldots, 9\}$ and the letters $\{a, A, b, B, c, C, \ldots, z, Z\}$ of the English language, including both lower-case and upper case letters, i.e., the letters are case sensitive.

a. How many strings of length 39 can be formed if repetition of symbols is *not* permitted?

$P(62, 39)$ OR $62 \cdot 61 \cdot 60 \cdot \ldots \cdot 25 \cdot 24$

b. How many strings of length 39 can be formed if repetition of symbols is permitted?

62^{39}

c. How many strings of length 39 can be formed using exactly twenty 5's, eight B's and eleven b's?

$\binom{39}{20, 8, 11}$ OR $\frac{39!}{20!\,8!\,11!}$

d. How many strings of length 39 can be formed using exactly twenty 5's, eight B's and eleven b's if the eight $B's$ are required to occur consecutively in the string?

$\binom{32}{20, 1, 11}$ OR $\frac{32!}{20!\,1!\,11!}$ Note: Treat block of B's as a single character

(6)

2. How many lattice paths from $(3, 5)$ to $(42, 69)$ pass through $(28, 52)$?

$\binom{72}{25}\binom{31}{14}$ Note: $(28-3 = 25) + (52-5 = 47) = 72$
$(42-28) + (69-52) = 14 + 17 = 31$

(15) 5×3

3. How many integer valued solutions to the following equations and inequalities:

a. $x_1 + x_2 + x_3 = 42$, all $x_i > 0$.

$\binom{41}{2}$ 41 gaps, choose 2

b. $x_1 + x_2 + x_3 = 42$, all $x_i \geq 0$.

$\binom{44}{2}$ Add three artificial's

c. $x_1 + x_2 + x_3 < 42$, all $x_i > 0$.

$\binom{41}{3}$ Add new variable $x_4 > 0$ and solve $x_1 + x_2 + x_3 + x_4 = 42$

d. $x_1 + x_2 + x_3 \leq 42$, all $x_i \geq 0$.

$\binom{45}{3}$ weaken to $x_4 \geq 0$. Now 4 artificials

e. $x_1 + x_2 + x_3 = 42$, all $x_i > 0$, $x_3 \geq 10$.

$\binom{32}{2}$ set aside 9 for x_3 33 remain.

(8)

4. Use the Euclidean algorithm to find $d = \gcd(231, 504)$.

$$231 \overline{)504} \quad \text{quotient } 2; \quad 462, \quad 42$$

$$42 \overline{)231} \quad \text{quotient } 5; \quad 210, \quad 21$$

$$21 \overline{)42} \quad \text{quotient } 2; \quad 42, \quad 0$$

$d = 21$ (last positive remainder)

(8)

5. Use your work in the preceding problem to find integers a and b so that $d = 231a + 504b$.

$$504 = 2 \cdot 231 + 42$$

$$231 = 5 \cdot 42 + 21$$

$$1 \cdot 504 - 2 \cdot 231 = 42$$

$$1 \cdot 231 - 5 \cdot 42 = 21$$

$$21 = 1 \cdot 231 - 5 \cdot 42$$

$$= 1 \cdot 231 - 5[1 \cdot 504 - 2 \cdot 231]$$

$$= 11 \cdot 231 - 5 \cdot 504$$

$a = 11 \quad b = -5$

(9)

6. For a positive integer n, let t_n count the number of ternary strings of length n that do not contain 001 as a substring. Note that $t_1 = 3$, $t_2 = 9$ and $t_3 = 26$. Develop a recurrence relation for t_n and use it to compute t_4, t_5 and t_6.

Consider last digit. If its a "0" or "2", then in front of it is a good sequence. On the other hand, if it's a "1", then in front of it is a good sequence ~~except~~ that it cannot end 00. This leads to

$$t_{n+1} = 2t_n + (t_n - t_{n-2})$$

$$= 3t_n - t_{n-2}$$

So

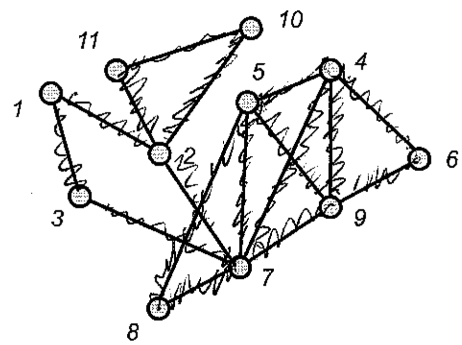
$$t_4 = 3t_3 - t_1 = 3 \cdot 26 - 3 = 78 - 3 = 75$$

$$t_5 = 3t_4 - t_2 = 3 \cdot 75 - 9 = 225 - 9 = 216$$

$$t_6 = 3t_5 - t_3 = 3 \cdot 216 - 26 = 648 - 26 = 622$$

7. Use the algorithm developed in class, with vertex 1 as root, to find an Euler circuit in the graph G shown below:

12

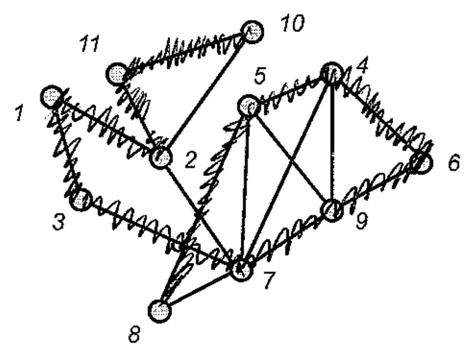


(1, 2, 7, 3, 1)
(2, 10, 11, 2)
(1, 2, 10, 11, 2, 7, 3, 1)
(7, 4, 5, 7, 8, 5, 9, 4, 6, 9, 7)

(1, 2, 10, 11, 2, 7, 4, 5, 7, 8, 5, 9, 4, 6, 9, 7, 3, 1)

8. Consider again the graph G from the preceding problem.

9
3x3



a. Show that there is a path starting at 10 and ending at 8 which visits each of the vertices, exactly once, along the way. You may answer this question either by listing the eleven vertices in a suitable order, or by darkening edges directly on the figure.

(10, 11, 2, 1, 3, 7, 9, 6, 4, 5, 8) other answers are possible

b. What is $\omega(G)$?

$\omega(G) = 4$ Note: $\{4, 5, 7, 9\}$ is a maximum clique

c. What is the chromatic number of the complete bipartite graph $K_{11,96}$?

$\chi(K_{11,96}) = 2$ Note: There are two parts. All vertices in the same part are colored the same.

21

9. True–False. Mark in the left margin.

F 1. $\binom{10}{3} = 160$. $\binom{10}{3} = \frac{10 \cdot \cancel{9}^{3} \cdot \cancel{8}^{4}}{\cancel{1 \cdot 2 \cdot 3}} = 10 \cdot 12 = 120$

F 2. $P(10,7) = 10 \cdot 9 \cdot 8 \cdot 7$. $P(10,7) = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$

F 3. The answer to question 1, part a, on this test is less than 1,000,000,000,000,000,000,000,000. $= 10^{21}$
Answer is product of 39 numbers, each bigger than 20

F 4. There is a graph G on 238 vertices with $\chi(G) = 17$ and $\omega(G) = 35$. $\chi(G) \geq \omega(G)$ for all G

T 5. All graphs with 1286 vertices and 5973 edges are non-planar. $5973 > 3 \cdot 1286 - 6$

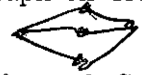
T 6. There is a hamiltonian graph on 684 vertices in which every vertex has degree 10.

T 7. Every connected graph on 684 vertices in which every vertex has degree 10 has an Euler circuit.

F 8. Every graph with 21 vertices and 231 edges is hamiltonian.

T 9. Every graph with 21 vertices in which every vertex has degree at least 11 is hamiltonian. $11 = \lceil \frac{21}{2} \rceil$ (Dirac)

T 10. If G is an interval graph, then $\chi(G) = \omega(G)$.

T 11. There is a planar graph on 458 vertices which is a homeomorph of the complete bipartite graph $K_{2,3}$.  Add vertices on edges

F 12. When $n \geq 3$, the shift graph S_n has $\binom{n}{2}$ edges and $\binom{n}{3}$ vertices. (backwards)

T 13. When $n \geq 3$, the maximum clique size of the shift graph S_n is given by $\omega(S_n) = 2$.

F 14. When $n \geq 3$, the chromatic number of the shift graph S_n is given by $\chi(S_n) = 3n - 6$. $\lceil \lg n \rceil$

F 15. Euler's formula asserts that if V, E and F count the number of vertices, edges and faces in a drawing (without crossings) of a connected planar graph, then $V - F + E = 2$. E and F reversed

T 16. The number of lattice paths from $(0,0)$ to (n,n) which do not pass through a point above the diagonal is the Catalan number $\binom{2n}{n}/(n+1)$.

T 17. Any modern computer can quickly add two 300 digit numbers.

T 18. Any modern computer can quickly multiply two 300 digit numbers.

F 19. Any modern computer can quickly test whether a 300 digit number is prime. I wish!!

T 20. Any modern computer can accept a file of 1000 positive integers, each at most 2000, and quickly determine whether 947 is one of the integers in the file.

T 21. Any modern computer can accept a file of 1000 positive integers, each at most 2000 and quickly determine whether there are two integers m and n in the file so that $m + n = 947$.

F 22. Merge Sort proceeds by: (1) splitting a sequence of length n into a planar shift graph and a connected multinomial coefficient; (2) extracting an Euler circuit; (3) forming the sum $\sum_{i=1}^{n} 3i - 6$; (4) producing a combinatorial proof of $2 = 1 + 1$; (5) repeating the previous part by induction; and finally (6) showing that interval graphs are homeomorphs of Catalan numbers. Just testing your sense of humor!!
Of course, this last question isn't graded.