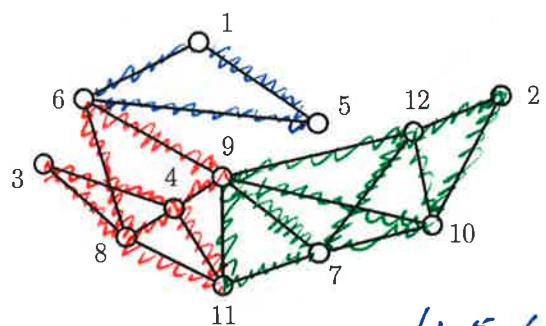
Solutions

Student Name and ID Number

MATH 3012 Quiz 2, October 28, 2014

10

1. In the space to the right, use the algorithm developed in class (always proceed to the lowest legal vertex) to find an Euler circuit in the graph G shown below (use node 1 as root):



~~(1, 5, 6, 1)~~
~~(6, 8, 3, 4, 8, 11, 7, 9~~, 6)
1, 5, ~~6, 8, 3, 4, 8, 11, 7, 9,~~ 6, 1
~~8, 11, 4, 9~~

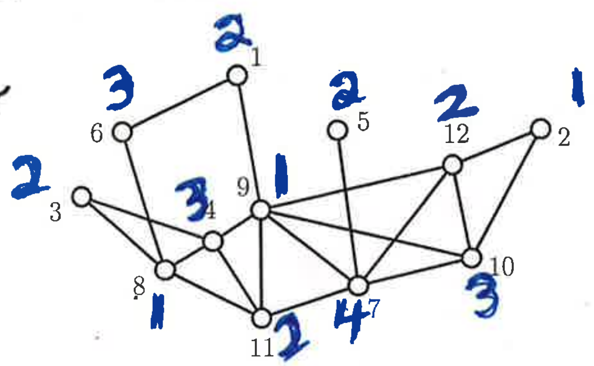
(1, 5, 6, 1) + 2
(6, 8, 3, 4, 8, 11, 4, 9, 6) + 2
(1, 5, 6, 8, 3, 4, 8, 11, 4, 9, 6, 1) + 2
(11, 7, 9, 10, 2, 12, 7, 10, 12, 9, 11) + 2

(1, 5, 6, 8, 3, 4, 8, 11, 7, 9, 10, 2, 12, 7, 10, 12, 9, 11, 4, 9, 6, 1) + 2

10

5 + 2

2. For the graph below,



(a) Find a maximal clique of size 2. 5, 7

(b) Find a maximal clique of size 3. {9, 10, 12}, {3, 4, 8}, {4, 8, 11}, ...

(c) Find a maximal clique of size 4. {7, 9, 10, 12}

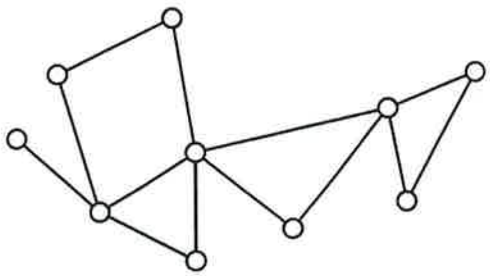
(d) Find an induced cycle of size 5. {1, 9, 4, 8, 6}

(e) Show that $\chi(G) \leq 4$ by producing a proper coloring using the elements of $\{1, 2, 3, 4\}$ as colors. You may write directly on the figure.

Page total 20

8

3. In the space to the right, verify Euler's formula for the following graph.



$V = 10$ +2

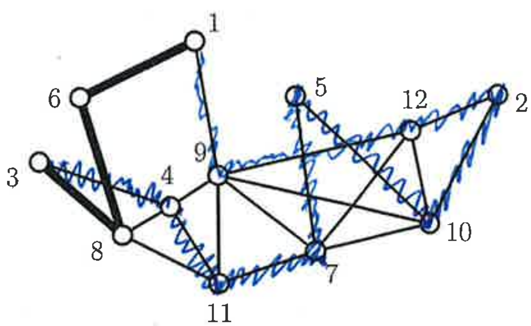
$E = 13$ +2

$F = 5$ +2

$V - E + F = 10 - 13 + 5 = 2$ +2 ✓

8

4. In the space to the right, explain why the following graph does not have an Euler circuit—*without* carrying out the algorithm from Problem 1. Then show that it has a hamiltonian cycle starting $(1, 6, 8, 3)$. You may either complete the sequence or darken the appropriate edges on the figure.



a. vertices 7 and 10 have odd degree.

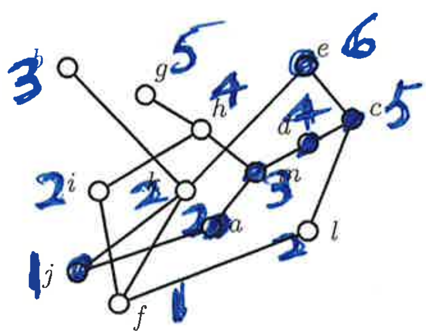
b. (1, 6, 8, 3, 4, 11, 7, 5, 10, 2, 12, 9, 1)

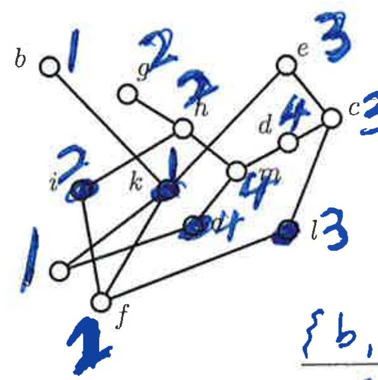
15

9×1 ×

2×3

5. Consider the poset shown below (two copies are shown).





1 each

(a) Find the set of maximal elements. {b, g, e}

(b) Find the set of minimal elements. {f, j}

(c) Find all points comparable with h. {g, i, f, m, a, j}

(d) Find all points incomparable with e. {b, g, h, i}.

(d) Find all points covered by h. {i, m}

(e) Find all points which cover h. {g}

page total 31

1 each

(f) Find a maximal chain of size 3. {b, k, j}, {b, k, f} {e, h, g} +1 more

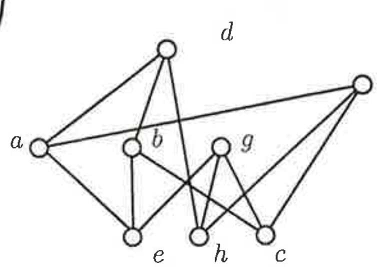
(g) Find a maximal antichain of size 2. {f, j} {f, m}

(h) This poset is not an interval order. Find, by inspection, four points which form a copy of $\mathbf{2}+\mathbf{2}$.
{b, k} {g, h} Many others

(3) (i) Recursively strip off the minimal elements and find the height h of the poset. Also find a partition of the poset into h antichains. You may provide your answer by labelling the points in the figure on the left with integers from $\{1, 2, \ldots, h\}$ so that all points labelled with the same integer form an antichain.
The height h is 6 and ~~f, k,~~ {j, a, m, d, c, e} is a maximum chain.

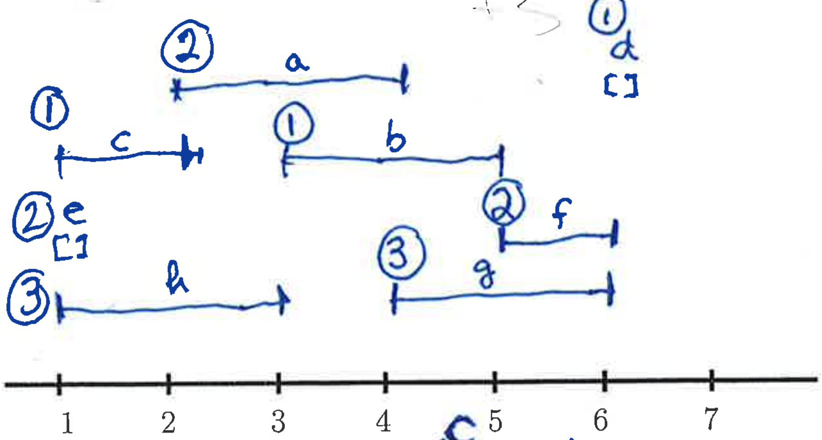
(3) (j) Find, by inspection, the width w of the poset. Also, find a partition of the poset into w chains. You may provide your answer by labelling the points in the figure on the right with integers from $\{1, 2, \ldots, w\}$ so that points labelled with the same integer form a chain.
The width w is 4 and {c, k, a, e} is a maximum antichain.

(10) **6.** Shown below is the diagram of an interval order. Use the algorithm taught in class to find an interval representation by computing the down-sets and up-sets in the space provided. Then use the First Fit coloring algorithm to find the width w and a partition of the poset into w chains. Also, find a maximum antichain.



2 $D(a) =$ e	4 $U(a) =$ d f
3 $D(b) =$ e c	5 $U(b) =$ d
1 $D(c) =$ ∅	2 $U(c) =$ b f d g
6 $D(d) =$ a b e c h	6 $U(d) =$ ∅
1 $D(e) =$ ∅	1 $U(e) =$ a d b f g
5 $D(f) =$ a e c h	6 $U(f) =$ ∅
4 $D(g) =$ e c h	6 $U(g) =$ ∅
1 $D(h) =$ ∅	3 $U(h) =$ d f g

+3 +3



The width w is 3 and {~~f~~ c, e, h} is a maximum antichain.

+2

{a, b, g} +2
{b, f, g}
{a, b, h} etc.

page total (10)

(8)
4×2

7. Let 2^{13} be the poset consisting of all subsets of $\{1, 2, 3, \ldots, 13\}$, ordered by inclusion.

a. What is the height of this poset? 14

b. What is the width of this poset? $\binom{13}{6}$ or $\binom{13}{7}$

c. How many maximal chains does the poset have? $13!$

d. How many maximal chains in this poset pass through the set $\{2, 4, 9, 11\}$? $4!\,9!$

8. Write the inclusion-exclusion formula for the number d_n of derangements of $\{1, 2, \ldots, n\}$. Then use this formula to calculate d_6.

(7)

$$d_n = \sum_{i=1}^{n} (-1)^i \binom{n}{i} (n-i)! \quad +4$$

$$d_6 = \binom{6}{0} 6! - \binom{6}{1} 5! + \binom{6}{2} 4! - \binom{6}{3} 3! + \binom{6}{4} 2! - \binom{6}{5} 1! + \binom{6}{6} 0! \quad +3$$

$$= \cancel{720} - \cancel{720} + 15 \cdot 24 - 20 \cdot 6 + 15 \cdot 2 - 6 \cdot 1 + 1 \cdot 1$$

9. Write the inclusion-exclusion formula for the number $S(n, m)$ of surjections from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, m\}$. Then use this formula to calculate $S(5, 3)$.

(7)

$$S(n, m) = \sum_{i=1}^{m} (-1)^i \binom{m}{i} (m-i)^n \quad +4$$

$$S(5,3) = \binom{3}{0} 3^5 - \binom{3}{1} 2^5 + \binom{3}{2} 1^5 - \binom{3}{3} 0^5$$

$$= 243 - 3 \cdot 32 + 3 \cdot 1 - 0 \quad +3$$

10. Note that $252 = 2^2 \cdot 7 \cdot 9$. Use this information and the inclusion-exclusion formula to determine $\phi(252)$, where ϕ is the Euler ϕ-function studied in class.

(7)

$$252 = 2^2 \cdot 7 \cdot 9$$
$$= 2^2 \cdot 3^2 \cdot 7 \quad +3 \text{ for } 9 = 3^2$$

$$\phi(252) = 252 \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{7}\right) \quad +4 \text{ formula}$$

$$= 252 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{6}{7}$$

$$= 2^2 \cdot 3^2 \cdot 7 \cdot \frac{1 \cdot 2 \cdot 6}{2 \cdot 3 \cdot 7} =$$

$$= 2^2 \cdot 3 \cdot 6$$

page total (29)

10

10×1

11. True–False. Mark in the left margin.

F 1. The number of labelled trees with vertex set $\{1, 2, \ldots, n\}$ is $\binom{n}{\lfloor n/2 \rfloor}$.

F 2. There is a graph with 21 vertices in which no two vertices have the same degree.

T 3. There is a connected graph with 21 vertices and 25 edges which has an euler circuit.

T 4. There is a connected graph with 21 vertices and 25 edges which does not have an euler circuit.

T 5. There is a connected graph with 21 vertices and 25 edges which has a hamiltonian cycle.

T 6. There is a connected graph with 21 vertices and 25 edges which does not have a hamiltonian cycle.

T 7. There is a connected graph with 21 vertices and 110 edges which is not hamiltonian.

F 8. There is a connected planar graph with 21 vertices and 110 edges.

F 9. There is a poset on 21 points having width 4 and height 3.

T 10. There is a poset on 21 points having width 10 and height 8.

FUN 11. The number of deranged surjections of a transitive hamiltonian trail is a planar function and and can be effectively computed using prime factors of even degree and odd behavior.

page total 10

Problem	Points	Pages
1	10	20
2	10	
3	8	31
4	8	
5	15	
6	10	10
7	8	29
8	7	
9	7	
10	7	
11	10	10
	100	100