Solutions

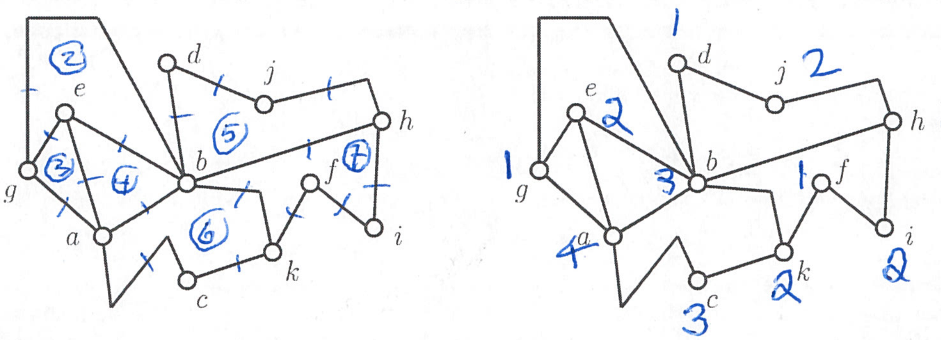
Student Name and ID Number

MATH 3012 Quiz 2, March 7, 2018 WTT

12

4x3

1. Two copies of a planar graph G with vertex set $\{a,b,c,d,e,f,g,h,i,j,k\}$ are shown below. Note that some of the edges in the drawing are straight lines while other edges are not.



a. Verify Euler's formula for the graph G. You may mark on the left copy of G if you find it convenient to do so.

$V - E + F = 2$

$11 - 16 + 7 = 2$ ✓

b. Find four vertices of G which form a clique of size 4. a, b, e, g

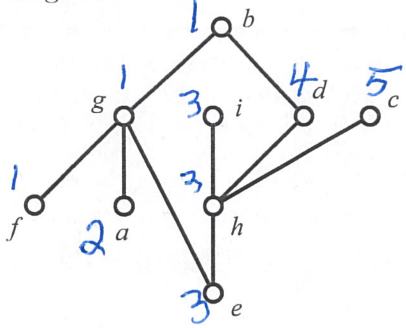
c. Show that $\chi(G) = \omega(G) = 4$ by indicating a 4-coloring of G on the right copy. Many ways to do this

d. Explain why G is not perfect by listing a sequence of vertices showing that G contains an induced cycle of size 5. b, h, i, f, k

9

3x3

2. Find by inspection the width w of the following poset and find a partition of the poset into w chains. Also find a maximum antichain. You may indicate the partition by writing directly on the diagram.



many correct ways

a. The width w is 5 and f, a, i, d, c is a maximum antichain.

b. This poset is not an interval order. Find by inspection four points which form a copy of $\mathbf{2}+\mathbf{2}$. ______.

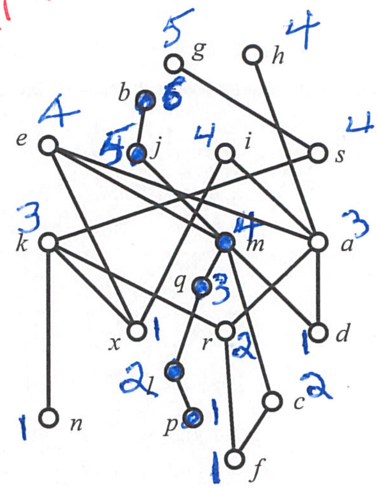
$\{g, f, i, h\}$

several other correct answers

12

7×1 + 5

3. Consider the following poset.



a. Find all points comparable to k. n, x, r, f, s, g

b. Find all points which cover k. s

c. Find all points which are covered by k. n, x, r

d. Find a maximal chain of size 2. e, x ALSO i, x

e. Find a maximal chain of size 3. h, a, d

f. Find the set of all maximal elements. e, b, g, i, h

g. Find the set of all minimal elements. n, x, p, f, d

h. Using the algorithm taught in class (recursively removing the set of minimal elements), find the height h of the poset and a partition of P into h antichains. Also find a maximum chain. You should indicate the partition by writing directly on the diagram, i.e., each element should be labeled with an integer from $\{1, 2, \ldots, h\}$.

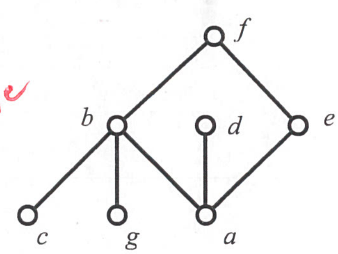
The height h is 6 (six) and b, j, m, q, l, p is a maximum chain.

13

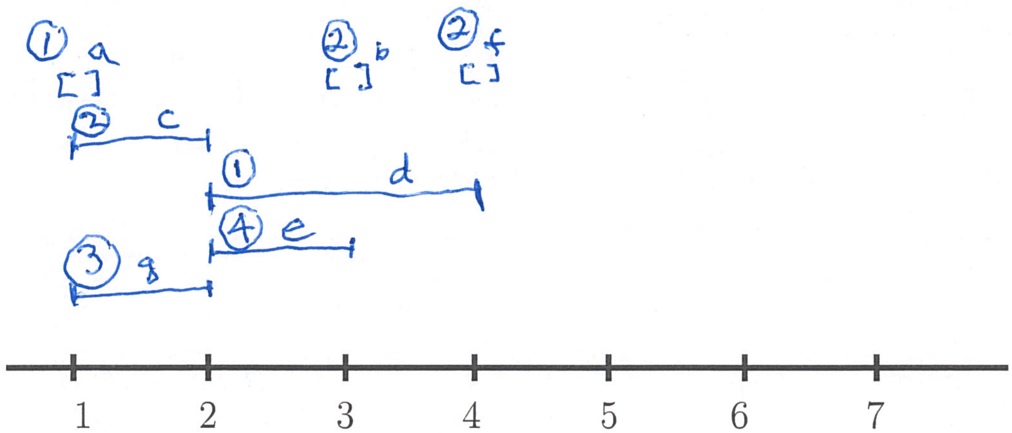
5+5+3

five

4. Shown below is the diagram of an interval order. Use the algorithm taught in class to find an interval representation by computing the down-sets and up-sets in the space provided. Then use the First Fit coloring algorithm to find the width w and a partition of the poset into w chains. Also, find a maximum antichain.



$D(a) =$ ∅	1	1	$U(a) =$	b d e f
$D(b) =$ a c g	3	3	$U(b) =$	f
$D(c) =$ ∅	1	2	$U(c) =$	b f
$D(d) =$ a	2	4	$U(d) =$	∅
$D(e) =$ a	2	3	$U(e) =$	f
$D(f) =$ b e a c g	4	4	$U(f) =$	∅
$D(g) =$ ∅	1	2	$U(g) =$	b f



The width w is 4 and c, g, d, e is a maximum antichain.

6

3+3

5.

a. Write in product form the generating function for the number of partitions of an integer n into parts, all of which are of odd size with no two parts having the same size.

$$f(x) = (1+x)(1+x^3)(1+x^5)(1+x^7)(1+x^9)(1+x^{11})\cdots$$

b. Write all the partitions of the integer 16 into parts, all of which are of odd size with no two parts having the same size.

$$\begin{aligned} 16 &= 15+1 \qquad = 7+5+3+1 \\ &= 13+3 \\ &= 11+5 \\ &= 9+7 \end{aligned}$$

8

4+3+1

found

6. Write the inclusion-exclusion formula for the number d_n of derangements of $\{1, 2, \ldots, n\}$. Then use your formula to find d_5.

$$d_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)!$$

$$\begin{aligned} d_5 &= \binom{5}{0}5! - \binom{5}{1}4! + \binom{5}{2}3! - \binom{5}{3}2! + \binom{5}{4}1! - \binom{5}{5}0! \\ &= 1\cdot 120 - 5\cdot 24 + 10\cdot 6 - 10\cdot 2 + 5\cdot 1 - 1\cdot 1 \\ &= 120 - 120 + 60 - 20 + 5 - 1 \\ &= 44 \end{aligned}$$

8

16
32
64
~~128~~
~~256~~

9
27
81
243
729

7. Write the formula for the number $S(n,m)$ for the number of surjections from $[n] = \{1, 2, \ldots, n\}$ to $[m] = \{1, 2, \ldots, m\}$. Then use your formula to find $S(6, 3)$.

$$S(n,m) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} (m-i)^n$$

$$\begin{aligned} S(6,3) &= \sum_{i=0}^{3} (-1)^i \binom{3}{i} (3-i)^6 \\ &= \binom{3}{0}3^6 - \binom{3}{1}2^6 + \binom{3}{2}1^6 - \binom{3}{3}0^6 \\ &= 729 - 3\cdot \cancel{256}\ 64 + 3 = 729 - 192 + 3 = 540 \end{aligned}$$

$729 - 192 = 537$

8

4+3+1

8. Write the inclusion-exclusion formula for $\phi(n)$ when $n = p_1^{m_1} \cdot p_2^{m_2} \cdot \cdots \cdot p_k^{m_k}$ where $p_1, p_2, \ldots, p_k$ are the primes which divide n without remainder. Then use your formula to find $\phi(n)$ when $n = 2^3 \cdot 5^2 \cdot 19$.

$$\varphi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_k}\right)$$

$$\begin{aligned} \varphi(2^3\cdot 5^2 \cdot 19) &= 2^3\cdot 5^2 \cdot 19\left(1-\frac{1}{2}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{19}\right) \\ &= 2^3 \cdot 5^2 \cdot 19 \cdot \frac{1}{2} \cdot \frac{4}{5} \cdot \frac{18}{19} \\ &= 2^4 \cdot 5 \cdot 18 \\ &= 1440 \end{aligned}$$

16
90
1440

For inclusion-exclusion
4 pts for general formula
3 for correct expression for specific problem and 1 pt for final answer.

8

9.

4×2

a. The height of the subset lattice $\mathbf{2}^{17}$ is: 18

b. The width of the subset lattice $\mathbf{2}^{17}$ is: $\binom{17}{8}$ OR $\binom{17}{9}$

c. The number of maximal chains in the subset lattice $\mathbf{2}^{17}$ is: $17!$

d. The number of maximal chains in the subset lattice $\mathbf{2}^{17}$ passing through 00011001010000000 is:

$4!\,13!$

16

10. True–False. Mark in the left margin.

16×1

T 1. There is a graph G with $\omega(G) = 2$ and $\chi(G) = 496$.

T 2. There is a graph G with $\omega(G) = 3$ and $\chi(G) = 496$.

F 3. There is a planar graph G with $\omega(G) = 2$ and $\chi(G) = 496$.

F 4. There is a perfect graph G with $\omega(G) = 2$ and $\chi(G) = 496$.

T 5. If $\chi(G) = 2$, then G is perfect.

F 6. If $\chi(G) = 3$, then G is perfect.

T 7. There is a graph G with 240 vertices and 998 edges such that $\chi(G) = \omega(G) = 2$.

T 8. There is a graph with 240 vertices and 1024 edges.

T 9. There is a perfect graph with 240 vertices and 1024 edges.

F 10. There is a planar graph with 240 vertices and 1024 edges.

F 11. There is a poset with 4215 points having width 79 and height 39.

T 12. There is a poset with 4215 points having width 97 and height 93.

F 13. When $n \geq 3$, the shift graph S_n contains a triangle.

F 14. When $n \geq 2$, the shift graph S_n has $\binom{n}{3}$ vertices.

F 15. When $n \geq 2$, the shift graph S_n has $\binom{n}{2}$ edges.

F 16. To test whether a graph G is an interval graph, we use a 2-phase algorithm. In the first phase, we test whether G is a cover graph. In the second phase, we test whether G has an Euler circuit.

17. (Just for fun!) The generating function of a perfect subset is symmetric when the Eulerian chain is comparable to the triangle-free coefficient of a Taylor series, unless the Dilworth problem for the rule of V's is NP-complete.