SIAM J. DISCRETE MATH.
Vol. 10, No. 4, pp. 515–528, November 1997

001

THE ORDER DIMENSION OF PLANAR MAPS*

GRAHAM R. BRIGHTWELL† AND WILLIAM T. TROTTER‡

Abstract. This is a sequel to a previous paper entitled *The Order Dimension of Convex Polytopes*, by the same authors [*SIAM J. Discrete Math.*, 6 (1993), pp. 230–245]. In that paper, we considered the poset $\mathbf{P_M}$ formed by taking the vertices, edges, and faces of a 3-connected planar map $\mathbf{M}$, ordered by inclusion, and showed that the order dimension of $\mathbf{P_M}$ is always equal to 4. In this paper, we show that if $\mathbf{M}$ is any planar map, then the order dimension of $\mathbf{P_M}$ is still at most 4.

Key words. partially ordered sets, dimension, planar maps, planar graphs, convex polytopes

AMS subject classifications. 06A07, 05C35

PII. S0895480192238561

1. Introduction.

In this paper, we are concerned with planar maps. We shall allow loops and multiple edges, and we always consider a fixed representation of a graph in the plane. More formally, given a multigraph $G = (V, E)$, a *plane drawing* D of G is a representation of G by points and arcs in $\mathbf{R}^2$ in which two edges meet only at common vertices. A *planar map* $\mathbf{M}$ is a pair (G, D) consisting of a multigraph and a plane drawing thereof. In what follows, we do not distinguish between a vertex (edge) of G and the corresponding point (arc) of $\mathbf{R}^2$.

Deleting the vertices and edges of a planar map $\mathbf{M}$ from the plane leaves several connected components whose closures are the *faces* of $\mathbf{M}$. The unique unbounded face is called the *exterior face*. For the purposes of this paper, it is not treated in any special way.

Given a planar map $\mathbf{M}$, the planar dual $\mathbf{M}^*$ is defined in the usual way, taking a vertex F^* for each face F of $\mathbf{M}$, and, for each edge e of $\mathbf{M}$, an edge e^* in $\mathbf{M}^*$ joining the vertices of $\mathbf{M}^*$ corresponding to the two faces separated by e in $\mathbf{M}$. (In the special case where the edge e is a bridge, the dual edge e^* is a loop on the dual of the unique face containing e.) Then each vertex v of $\mathbf{M}$ corresponds to a face v^* in $\mathbf{M}^*$. If $\mathbf{M}$ is connected, then $\mathbf{M}^{**}$ is isomorphic to $\mathbf{M}$.

For a planar map $\mathbf{M}$, we form a poset $\mathbf{P_M}$ by taking the vertices, edges, and faces of $\mathbf{M}$ (including the exterior face), ordered by inclusion. See Figure 1.1 for an example of a planar map $\mathbf{M}$ and its associated poset $\mathbf{P_M}$. Let us note immediately that, if $\mathbf{M}$ is connected, the poset $\mathbf{P_{M^*}}$ associated with the dual map is just the dual poset $(\mathbf{P_M})^*$ (i.e., the set of vertices, edges, and faces ordered by reverse inclusion).

The *order dimension* $\dim(\mathbf{P})$ of a partial order $\mathbf{P}$ is the smallest number t such that $\mathbf{P}$ is the intersection of t linear orders on the same vertex set. The following result was proved in [1], answering a question of Reuter [3].

THEOREM 1.1. *For every 3-connected planar map $\mathbf{M}$, $\dim(\mathbf{P_M}) = 4$.* □

This result is to be compared with one due to Schnyder [4]: if G is any graph and $\mathbf{P}(G)$ is the poset formed from the vertices and edges of G, ordered by inclusion, then $\dim(\mathbf{P}(G)) \leq 3$ iff G is planar. If G is planar, and $\mathbf{M}$ is a map with underlying

*Received by the editors October 9, 1992; accepted for publication July 19, 1996.
http://www.siam.org/journals/sidma/10-4/23856.html

†Department of Mathematics, London School of Economics, Houghton Street, London WC2A 2AE, U. K. (g.r.brightwell@lse.ac.uk).

‡Department of Mathematics, Arizona State University, Tempe, AZ 82587-1804 (trotter@asu.edu). The research of this author was supported in part by the Office of Naval Research.

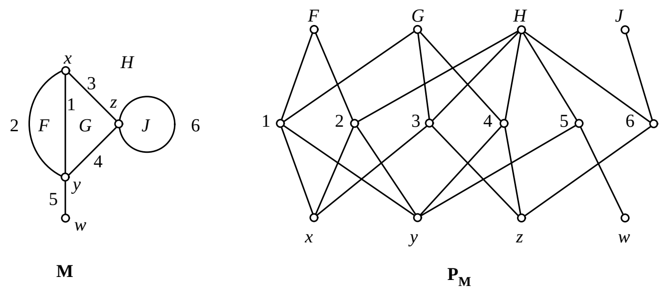

FIG. 1.1. *A planar map* $\mathbf{M}$ *and the poset* $\mathbf{P}_{\mathbf{M}}$

graph G, then $\mathbf{P}(G)$ is an induced subposet of our poset $\mathbf{P}_{\mathbf{M}}$. Thus, although we do not refer to it again explicitly, Schnyder's work underpins much of what we do in this paper.

One reason for restricting attention to 3-connected planar maps in [1] was the connection with convex polytopes in $\mathbf{R}^3$: each convex polytope gives rise to a 3-connected planar map $\mathbf{M}$ and the poset $\mathbf{P}_{\mathbf{M}}$ corresponds to the set of vertices, edges, and faces of the polytope, ordered by inclusion.

The main purpose of this paper is to prove the following result, extending Theorem 1.1 to general planar maps.

THEOREM 1.2. *Let* $\mathbf{M}$ *be a planar map, and let* $\mathbf{P}_{\mathbf{M}}$ *be the poset of all vertices, edges, and faces of* $\mathbf{M}$ *ordered by inclusion. Then* $\dim(\mathbf{P}_{\mathbf{M}}) \leq 4$.

For more information as to the origin of the problem, see [1], Reuter [3], or Schnyder [4].

In the course of proving Theorem 1.2, we shall use a result (Theorem 3.2) that is slightly stronger than Theorem 1.2 itself as the base case for an induction argument. However the machinery developed in [1] is used only in the proof of Theorem 3.2.

Before we begin, we need a few concepts from the theory of order dimension. For a comprehensive treatment of dimension theory for finite posets, we refer the reader to the monograph [6]. Other sources include the survey articles [2] and [5] and our previous paper [1]. Given a partial order $\mathbf{P}$, a set $\mathcal{R} = \{L_1, \ldots, L_t\}$ of linear extensions of $\mathbf{P}$ is called a *realizer* of $\mathbf{P}$ if the intersection of the L_i is exactly $\mathbf{P}$. Thus the order dimension of $\mathbf{P}$ is the minimum cardinality of a realizer.

An ordered pair (a, b) of elements of a partial order $\mathbf{P}$ is called a *critical pair* if the following three conditions hold:

(i) a and b are incomparable;
(ii) if $c < a$ in $\mathbf{P}$, then $c < b$; and
(iii) if $b < d$ in $\mathbf{P}$, then $a < d$.

An ordered pair (a, b) of elements of $\mathbf{P}$ is said to be *reversed* by a linear extension L if $b < a$ in L. It is fairly easy to see that a set $\{L_1, \ldots, L_t\}$ of linear extensions of $\mathbf{P}$ is a realizer if and only if every critical pair is reversed by some L_i.

If F is a face of $\mathbf{M}$ and x is a vertex not on F, then the pair (x, F) is a critical pair. We call this a *vertex-face critical pair* and extend the terminology in the obvious way. If all critical pairs of $\mathbf{P}_{\mathbf{M}}$ are of this vertex-face type, we say that $\mathbf{M}$ is *well formed*. It is easy to see that every 3-connected planar map (with no loops or multiple edges) is well formed.

For a planar map $\mathbf{M}$, we define another partial order $\mathbf{Q}_{\mathbf{M}}$ by taking just the vertices and faces of $\mathbf{M}$, ordered by inclusion. (Figure 1.2 shows the poset $\mathbf{Q}_{\mathbf{M}}$ for the map $\mathbf{M}$ in Figure 1.1.) Evidently $\mathbf{Q_M}$ is an induced subposet of $\mathbf{P_M}$, and so $\dim(\mathbf{Q_M}) \leq \dim(\mathbf{P_M})$. The reverse inequality is not true in general, but it does hold whenever $\mathbf{M}$ is well formed.

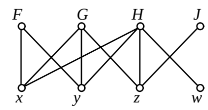

FIG. 1.2. *The poset* $\mathbf{Q_M}$

LEMMA 1.3. *Suppose that* $\mathbf{M}$ *is well formed. Then* $\dim(\mathbf{P_M}) = \dim(\mathbf{Q_M})$.

Proof. We have seen that $\dim(\mathbf{P_M}) \geq \dim(\mathbf{Q_M})$. Conversely, given a realizer $\{L_1, \ldots, L_t\}$ of $\mathbf{Q_M}$, we can insert the edges of $\mathbf{M}$ into each linear extension L_i in a way consistent with $\mathbf{P_M}$: this then gives a realizer of $\mathbf{P_M}$, since the critical pairs of $\mathbf{P_M}$ are all of vertex-face type and so are reversed by some L_i. ▯

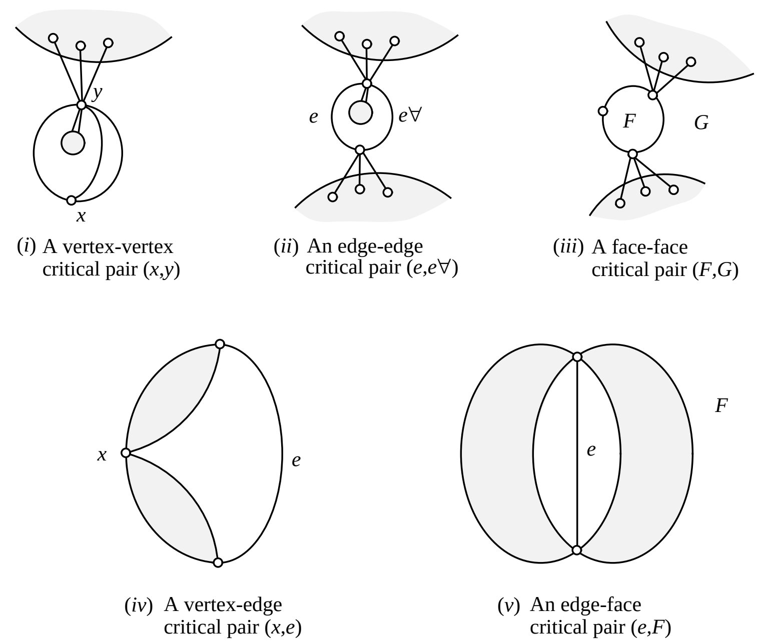

FIG. 1.3. *Examples of critical pairs.*

For a general planar map $\mathbf{M}$, the poset $\mathbf{P_M}$ may have vertex-vertex, edge-edge, and face-face critical pairs, but only if $\mathbf{M}$ is not 2-connected: see Figure 1.3(i)–(iii). However, $\mathbf{Q_M}$ can have vertex-vertex or face-face critical pairs even if $\mathbf{M}$ is 2-connected; for instance, if x is a vertex of degree 2 with distinct neighbors y and z, then (x, y) and (x, z) are critical pairs in $\mathbf{Q_M}$.

If (e, e') is an edge-edge critical pair in $\mathbf{P_M}$, the two edges must share the same endpoints and separate the same faces, as in Figure 1.3(ii). This makes edge-edge critical pairs very easy to deal with: given a set $\{L_1, L_2, \ldots, L_k\}$ of linear extensions reversing all other critical pairs, we move e' to the place immediately above e in L_1, and to the place immediately below e in all the other L_i. This yields a realizer. Thus we may effectively ignore edge-edge critical pairs.

Even if $\mathbf{M}$ is 2-connected, $\mathbf{P_M}$ may have vertex-edge or edge-face critical pairs. See Figure 1.3(iv) and (v) for examples. If e is an edge in such a critical pair, we call e a *critical edge*. The following trivial observation will be useful later.

LEMMA 1.4. *Let e be an edge of a planar map $\mathbf{M}$. Then e cannot be in both a vertex-edge and an edge-face critical pair.* □

Before we begin the proof of Theorem 3.2, we must clarify what we mean by k-connectivity for planar maps. The definition we use is not quite the usual one, since it is appropriate for the concept to be invariant under duality. For instance, the map in Figure 1.4 should not be 3-connected, since its dual isn't 3-connected.

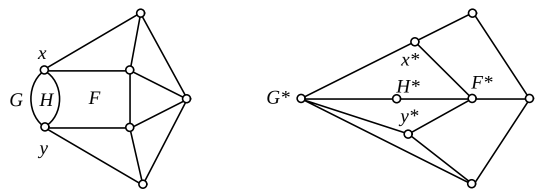

FIG. 1.4. *A map that is not 3-connected and its dual.*

The approach we adopt here is to define the connectivity of a planar map $\mathbf{M}$ to be the minimum of the connectivities of the underlying graphs of $\mathbf{M}$ and $\mathbf{M}^*$. Note that at least one of these graphs always contains a vertex of degree at most 3, so the only 4-connected maps are those with underlying graph K_4.

With the exception of a few graphs with at most three vertices, we have the following alternative characterizations. A map has connectivity 0 iff it is disconnected, connectivity 1 iff it is connected and has a cutvertex, and connectivity 2 iff either its underlying graph has connectivity 2 or it has a double edge, as in Figure 1.4.

If a map $\mathbf{M}$ with at least four vertices has connectivity exactly 2, then it has a pair $\{x, y\}$ of vertices and a pair $\{F, G\}$ of faces such that $\mathbf{R}^2 - (F \cup G \cup \{x, y\})$ falls into two components, neither of which is a single edge. We call $\{x, y, F, G\}$ a *separating system*. For instance, in Figure 1.4, $\{x, y, F, G\}$ is a separating system.

We shall approach Theorem 3.2 via the following intermediate result.

LEMMA 1.5. *Let $\mathbf{M}$ be a 2-connected planar map. Then* $\dim(\mathbf{Q_M}) \leq 4$.

The next section is devoted to the deduction of Theorem 3.2 from Lemma 1.5. Then in section 3 we prove Lemma 1.5. The basic idea involves modifying and combining families of linear extensions given to us from Theorem 1.1. However, the following observation gives some indication of the fundamental difference between the 3-connected case and the general case we are considering here.

For a 3-connected map $\mathbf{M}$, the poset $\mathbf{Q_M}$ is 4-irreducible, as shown in section 6 of [1]. Indeed, the proof of Theorem 1.1 was very much geared to proving that $\mathbf{Q_M}$ is "almost 3-dimensional": producing three linear extensions that are almost a realizer. But the poset $\mathbf{Q_M}$ for the map $\mathbf{M}$ in Figure 1.5 is not 4-irreducible; each critical pair (x_i, F_i) must be reversed by a different linear extension, so $\mathbf{Q_M}$ minus the outside face still has dimension 4. Thus, to prove Lemma 1.5 we shall have to make full use of the fact that we have four linear extentions to work with.

2. Reduction to the 2-connected case. We shall prove the following result, which clearly combines with Lemma 1.5 to give Theorem 3.2.

LEMMA 2.1. *If $\mathbf{M}$ is a planar map, then there exists a 2-connected planar map $\mathbf{M}_0$ such that* $\dim(\mathbf{P_M}) \leq \dim(\mathbf{Q_{M_0}})$.

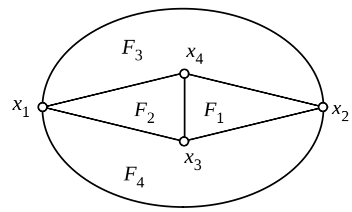

FIG. 1.5. *A map* $\mathbf{M}$ *for which* $\mathbf{Q_M}$ *is not* 4*-irreducible.*

Proof. If the map $\mathbf{M}$ is well formed and 2-connected, the result is immediate by Lemma 1.3. Thus, we shall consider in turn each of the ways in which $\mathbf{M}$ may fail to be 2-connected and well formed.

Our approach will be to construct a sequence of intermediate maps $\mathbf{M}_i$ from $\mathbf{M}$ such that a realizer of $\mathbf{P}_{\mathbf{M}_i}$ or of $\mathbf{Q}_{\mathbf{M}_i}$ can be converted into a realizer of $\mathbf{P_M}$.

We illustrate the process by showing in Figure 2.1 the sequence $\mathbf{M}_i$ of maps generated by starting from the map $\mathbf{M}$ with two vertices and one loop.

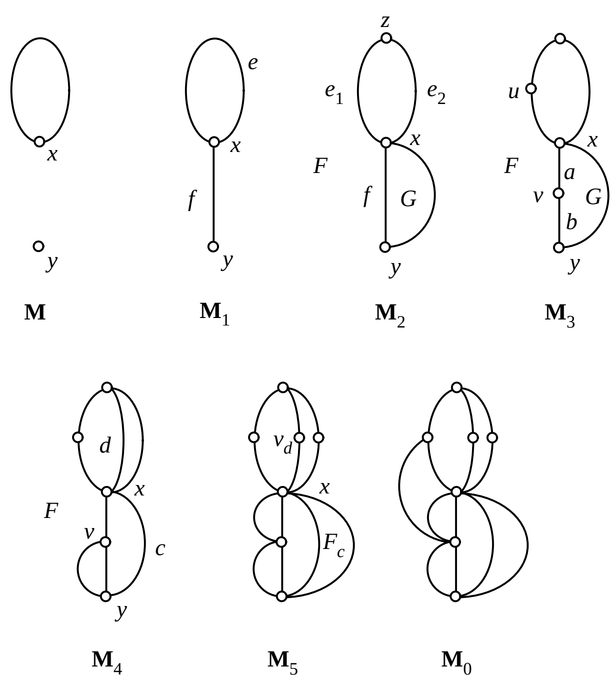

FIG. 2.1. *The proof of Lemma* 2.1.

(1) Making M connected. Given a planar map $\mathbf{M}$, we construct a connected map $\mathbf{M}_1$ from $\mathbf{M}$ by adding bridges between components as necessary. Clearly $\mathbf{P_M}$ is an induced subposet of $\mathbf{P}_{\mathbf{M}_1}$, so $\dim(\mathbf{P_M}) \leq \dim(\mathbf{P}_{\mathbf{M}_1})$.

(2) Destroying loops and vertices of degree 1. Suppose that, as in Figure 2.1, there is a loop e on a vertex x in $\mathbf{M}_1$. In this case, we form $\mathbf{M}_1'$ by subdividing e; i.e., we replace e by a vertex z and a pair of edges e_1 and e_2 joining x to z. Identifying e with e_1, we see that $\mathbf{P}_{\mathbf{M}_1}$ is an induced subposet of $\mathbf{P}_{\mathbf{M'}_1}$, and thus $\dim(\mathbf{P}_{\mathbf{M}_1}) \leq \dim(\mathbf{P}_{\mathbf{M'}_1})$.

By duality, we can also deal with the case where $\mathbf{M}_1$ has a vertex of degree 1. Note that the dual operation to subdividing an edge is that of duplicating an edge: replacing an edge f from x to y by two such edges surrounding a new face.

By repeating the process as often as necessary, we obtain a connected map $\mathbf{M}_2$ with no loops or vertices of degree 1 such that $\dim(\mathbf{P}_{\mathbf{M}_1}) \leq \dim(\mathbf{P}_{\mathbf{M}_2})$.

(3) Destroying vertex-vertex and face-face critical pairs. Suppose that, again as in Figure 2.1, there is a vertex-vertex critical pair (y, x) in $\mathbf{M}_2$. Then all the edges including y have x as their other endpoint. Choose one such edge e' and subdivide it with a vertex v, introducing new edges a, between x and v, and b, between v and y, in place of e'. This operation decreases the number of vertex-vertex critical pairs without introducing any extra face-face critical pairs. Let F and G be the two faces separated by e'. Call the new map $\mathbf{M}_2'$.

Suppose $\{L_1, \dots, L_t\}$ is a realizer of $\mathbf{P}_{\mathbf{M'}_2}$. For each $i = 1, \dots, t$, we construct a linear extension L_i' of $\mathbf{P}_{\mathbf{M}_2}$ from L_i as follows. We insert e' immediately above the highest of a, b, v in L_i; then we delete a, b, and v from the ordering. This is certainly a linear extension, since e' is placed above x, y and below F, G.

We claim that $\{L_1', \dots, L_t'\}$ is a realizer of $\mathbf{P}_{\mathbf{M}_2}$. When restricted to $\mathbf{P}_{\mathbf{M}_2} - e'$, the intersection of the L_i' is the same as the intersection of the L_i, so it remains to check that all critical pairs involving e' are reversed. Clearly e' is not in any vertex-edge critical pairs and, as mentioned in section 1, edge-edge critical pairs can be ignored. If (e, H) is an edge-face critical pair, then (v, H) is reversed in some L_k, and hence (e, H) is reversed in L_k'.

Thus $\dim(\mathbf{P}_{\mathbf{M}_2}) \leq \dim(\mathbf{P}_{\mathbf{M'}_2})$. Proceeding in this manner we can remove all the vertex-vertex critical pairs. Thus we construct a map $\mathbf{M}_3$ with no critical pairs of this type such that $\dim(\mathbf{P}_{\mathbf{M}_2}) \leq \dim(\mathbf{P}_{\mathbf{M}_3})$.

Using the dual case of the above argument, we can next find a map $\mathbf{M}_4$ with no critical pairs of either vertex-vertex or face-face type such that $\dim(\mathbf{P}_{\mathbf{M}_3}) \leq \dim(\mathbf{P}_{\mathbf{M}_4})$. For instance, in the map $\mathbf{M}_3$ of Figure 2.1, (G, F) is a critical pair, which is destroyed by duplicating the edge b.

(4) Destroying vertex-edge and edge-face critical pairs. Our approach to critical pairs of these types will be slightly different. We shall deal with all the vertex-edge and edge-face critical pairs in one step, forming an auxiliary map $\mathbf{M}_5$ such that $\dim(\mathbf{P}_{\mathbf{M}_4}) \leq \dim(\mathbf{Q}_{\mathbf{M}_5})$.

Recall from Lemma 1.4 that no edge is in both a vertex-edge and an edge-face critical pair. We form $\mathbf{M}_5$ as follows. For every edge e, say between x and y, of $\mathbf{M}_4$ which is in a vertex-edge critical pair, replace e by a double edge from x to y, and call the face between the two edges F_e. For every edge e of $\mathbf{M}_4$ in an edge-face critical pair, subdivide e with a vertex v_e. (The idea is that the new element F_e or v_e will represent the critical edge e in $\mathbf{M}_5$.) For instance, in the map $\mathbf{M}_4$ of Figure 2.1, (v, c) and (d, F) are critical pairs of $\mathbf{P}_{\mathbf{M}_4}$, so c is duplicated to produce a face F_c, and d is subdivided by a vertex v_d.

Let $\{L_1, \dots, L_t\}$ be a realizer of $\mathbf{Q}_{\mathbf{M}_5}$. From each L_i, we construct a linear extension L_i' of $\mathbf{P}_{\mathbf{M}_4}$ as follows. We start from L_i, which includes all vertices and faces of $\mathbf{P}_{\mathbf{M}_4}$, and insert the edges according to the following rules. First, noncritical edges of $\mathbf{M}_4$ are inserted anywhere consistent with the order $\mathbf{P}_{\mathbf{M}_4}$. Next, if e is a critical edge in a vertex-edge critical pair, with e separating faces F and G in $\mathbf{M}_4$, say, then e is inserted just below the lowest of F, G, and F_e in L_i. Similarly, if e is an edge in an edge-face critical pair, with e joining x and y, then e is inserted into L_i just above the highest of x, y, and v_e. Finally the auxiliary vertices and faces v_e and

F_e are deleted from the linear extension.

The L_i thus constructed are clearly linear extensions of $\mathbf{P}_{\mathbf{M}_4}$. It may be that some edge-edge critical pairs are not reversed: if this is the case, we alter the L_i so that they are, as in section 1. Certainly all vertex-face critical pairs in $\mathbf{P}_{\mathbf{M}_4}$ are reversed by some L_i. It remains to be shown that all vertex-edge and edge-face critical pairs are reversed. The two cases are dual, so we need only consider a vertex-edge critical pair (v, e) of $\mathbf{P}_{\mathbf{M}_4}$. For such a pair, we have an auxiliary face F_e, and the pair (v, F_e) is reversed in some L_k. Hence (v, e) is reversed in L'_k.

Thus all critical pairs of $\mathbf{P}_{\mathbf{M}_4}$ are reversed by some L'_i, and so $\{L'_1, \ldots, L'_t\}$ is a realizer of $\mathbf{P}_{\mathbf{M}_4}$, as required.

(5) Making the map 2-connected. We proceed by reducing the number of blocks of the underlying graph of $\mathbf{M}_5$ to 1, noting that no endblock is a single edge or a loop. If $\mathbf{M}_5$ is not 2-connected, let x be any cutvertex of the underlying graph, and let F be a face with x occurring at least twice on its boundary, as in Figure 2.1. The sequence of vertices encountered by travelling around the boundary of F thus includes x (indeed, more than once): let u and v be the vertices just before and after x in one such encounter. Form $\mathbf{M}'_5$ by joining y and z by an edge, thus decreasing the number of blocks. Clearly $\mathbf{Q}_{\mathbf{M'}_5} = \mathbf{Q}_{\mathbf{M}_5}$. Repeating as necessary, we end with a 2-connected map $\mathbf{M}_0$ such that $\mathbf{Q}_{\mathbf{M}_0} = \mathbf{Q}_{\mathbf{M}_5}$.

Combining all the steps, we see that $\dim(\mathbf{P}_\mathbf{M}) \leq \dim(\mathbf{Q}_{\mathbf{M}_0})$, as desired. □

3. Proof of Lemma 1.5. Throughout this section, e will be a distinguished edge in a 2-connected planar map $\mathbf{M}$. The endpoints of e will always be denoted x and y, and the faces separated by e by F and G.

For a planar map $\mathbf{M}$ with distinguished edge e, we say that a realizer $\mathcal{R}$ of $\mathbf{Q}_\mathbf{M}$ is an *e-realizer* if it has order 4, and the four linear extensions in $\mathcal{R}$ can be labelled L_1, L_2, L_3, L_4 so as to satisfy the following conditions:

(a) x is the highest vertex in L_1,
(b) y is the highest vertex in L_2,
(c) F is the lowest face in L_3, and
(d) G is the lowest face in L_4.

We shall prove the following result, which is stronger than Lemma 1.5.

Theorem 3.1. *Let* $\mathbf{M}$ *be a 2-connected planar map, and let e be an edge of* $\mathbf{M}$. *Then there is an e-realizer of* $\mathbf{Q}_\mathbf{M}$.

One technical problem we have to deal with is that $\mathbf{Q}_\mathbf{M}$ will in general have vertex-vertex and face-face critical pairs. In fact, a glance at the proof of Lemma 2.1 shows that we can ignore these: to prove Theorem 1.2 it is enough to show that, for every 2-connected map $\mathbf{M}$, there is a set of four linear extensions of $\mathbf{Q}_\mathbf{M}$ reversing every vertex-face critical pair of $\mathbf{Q}_\mathbf{M}$. However, it involves essentially no extra work to prove Theorem 3.1 as it stands, since the constructions we shall give do yield realizers of $\mathbf{Q}_\mathbf{M}$.

Let us first see that Theorem 3.1 holds if $\mathbf{M}$ is 3-connected. We use the notation and techniques of [1]. The reader who does not have that paper at hand may rest assured that the proof is a straightforward application of the methods developed there.

Theorem 3.2. *Let* $\mathbf{M}$ *be a 3-connected planar map, and let e be an edge of* $\mathbf{M}$. *Then there is an e-realizer of* $\mathbf{Q}_\mathbf{M}$.

Proof. Arrange for G to be the outside face, with x and y two vertices of a triad $(v_1 = x, v_2 = y, v_3)$, and apply the construction of [1] with this triad to obtain a realizer consisting of four linear extensions L_1, L_2, L_3, and L_4, as in [1]. Certainly G is the lowest face in the fourth linear extension L_4. Also, x is the highest vertex in L_1, since it is the only vertex w with $S(w,1)$ equal to the whole of $\mathbf{R}^2 - \mathrm{int}(G)$. Similarly, y is the highest vertex in L_2.

The face F is contained in $S(w,3)$ for every vertex w except for x and y. Thus if z is any vertex on F and u is any vertex not on F, we have $S(z,3) \subseteq S(u,3)$. If $S(z,3) = S(u,3)$, then either (F,y) witnesses $(z,u) \in \mathcal{R}_3'$ or (F,x) witnesses $(z,u) \in \mathcal{L}_3'$. In any case, (z,u) in Q_3'. Thus in fact F lies below all vertices not on F in L_3 and is certainly the lowest face in that order. Therefore the set $\{L_1, L_2, L_3, L_4\}$ is an e-realizer. □

We make one more observation before the proof of Theorem 3.1. Let $\mathcal{R}$ be an e-realizer of a planar map $\mathbf{M}$. We call $\mathcal{R}$ a *strong e-realizer* if its four linear extensions can be labelled L_1, L_2, L_3, L_4 so that, in addition to properties (a) to (d) above, we have that

(e) y is the lowest element of L_1, and F and G the two highest elements;
(f) x is the lowest element of L_2, and F and G the two highest elements;
(g) x and y are the two lowest elements of L_3, and G the highest element; and
(h) x and y are the two lowest elements of L_4, and F the highest element.

LEMMA 3.3. *Let e be a distinguished edge in a 2-connected planar map $\mathbf{M}$. If $\mathbf{Q_M}$ has an e-realizer, then it has a strong e-realizer.*

Proof. Let (L_1, L_2, L_3, L_4) be a realizer satisfying (a) through (d). If there are any faces above x in L_1 which do not contain x, they can be moved to a position in L_1 below x but above all other vertices. The altered set of linear extensions is clearly still an e-realizer of $\mathbf{Q_M}$. Thus we may assume that all critical pairs involving x are reversed in L_1.

Having made this assumption, we may then also suppose that x is the lowest element in all of the other three linear extensions: if not, it can be moved to the bottom, since the only critical pairs this affects are those involving x.

Proceeding in a similar way, we can alter the linear extensions so as to move y, F, and G to the positions required by (e) through (h). □

Proof of Theorem 3.1. We proceed by induction on the number of edges of $\mathbf{M}$. It is easily checked that the result is true for all 2-connected planar maps with at most, say, 4 edges.

Let $\mathbf{M}$ be a 2-connected planar map with $m \geq 5$ edges, and suppose that the result is true for all 2-connected maps with fewer than m edges. Let e be an edge of $\mathbf{M}$.

If $\mathbf{M}$ is 3-connected, then $\dim(\mathbf{Q_M}) \leq 4$ by Theorem 3.2. Suppose then that $\mathbf{M}$ is not 3-connected.

The dual map $\mathbf{M}^*$ of $\mathbf{M}$ is also 2-connected. Let e^* be the edge of $\mathbf{M}^*$ corresponding to e, and suppose that there is an e^*-realizer $\{L_1, \ldots, L_4\}$ of $\mathbf{Q_{M^*}}$. Then the set $\{L_1^*, \ldots, L_4^*\}$ of reverse linear orders provides an e-realizer of $\mathbf{Q_M}$. In other words it would suffice to prove the result for $\mathbf{M}^*$ and e^* instead of for $\mathbf{M}$ and e.

We split the argument into two cases, according to whether or not e is a critical edge in $\mathbf{M}$. In both cases, our task is to construct either an e-realizer of $\mathbf{Q_M}$ or an e^*-realizer of $\mathbf{Q_{M^*}}$.

(A) *e* is a critical edge. Suppose that (e,H) is an edge-face critical pair: if instead e is in a vertex-edge critical pair, then we work instead in the dual.

Removal of x, y, e, and H from the plane leaves two components, one containing F and the other G. Let $\mathbf{M}_1$ be the submap of $\mathbf{M}$ specified by the edges in the F-component together with e; and let $\mathbf{M}_2$ be the submap specified by e and the edges in the G-component. In both cases, let H stand for the exterior face. See Figure 3.1.

Thus the elements in common between $\mathbf{Q}_{\mathbf{M}_1}$ and $\mathbf{Q}_{\mathbf{M}_2}$ are just x, y, and H; and there are no relations in $\mathbf{Q}_{\mathbf{M}}$ between an element of $\mathbf{Q}_{\mathbf{M}_1}$ and an element of $\mathbf{Q}_{\mathbf{M}_2}$ except those involving x, y, or H. Also, if (α, β) is a vertex-vertex or face-face critical pair, then α and β must either both be in $\mathbf{Q}_{\mathbf{M}_1}$ or both be in $\mathbf{Q}_{\mathbf{M}_2}$, except that (F, G) or (G, F) could be a critical pair.

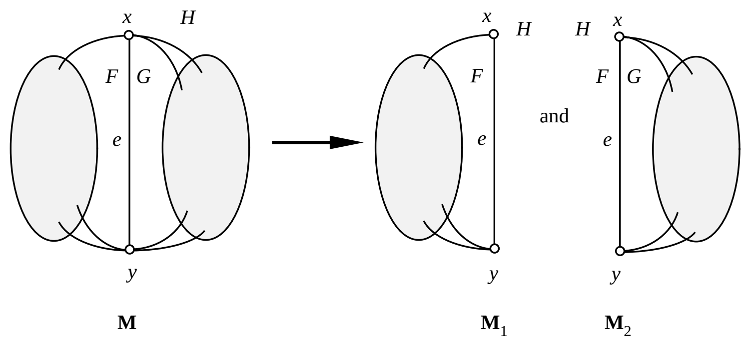

FIG. 3.1. *Splitting* $\mathbf{M}$ *into* $\mathbf{M_1}$ *and* $\mathbf{M_2}$.

Now $\mathbf{M}_1$ and $\mathbf{M}_2$ both have fewer edges than $\mathbf{M}$, so we can find an e-realizer for each map. To be more specific, we can find a realizer $(L_1^1, L_2^1, L_3^1, L_4^1)$ of $\mathbf{Q}_{\mathbf{M}_1}$ and a realizer $(L_1^2, L_2^2, L_3^2, L_4^2)$ of $\mathbf{Q}_{\mathbf{M}_2}$ satisfying the following:

(i) x is the highest vertex in both L_1^1 and L_1^2,
(ii) y is the highest vertex in both L_2^1 and L_2^2,
(iii) F is the lowest face in L_3^1,
(iv) G is the lowest face in L_4^2, and
(v) H is the lowest face in both L_3^2 and L_4^1.

By Lemma 3.3, we may also take these two realizers to be strong e-realizers, so in particular we may assume that H is the highest element in both L_3^1 and L_4^2 and that x and y are the lowest elements in L_3^2 and L_4^1.

Now, for $j = 1, \dots, 4$, we combine the linear extensions L_j^1 and L_j^2 to form a linear extension L_j of $\mathbf{Q}_{\mathbf{M}}$ as follows. For $j = 1, 2$, we form L_j in any way such that the restriction of L_j to the elements of $\mathbf{Q}_{\mathbf{M}_i}$ is L_j^i, for $i = 1, 2$. Hence x is the highest vertex in L_1, and y the highest in L_2.

For L_3, we essentially put L_3^2 above L_3^1. To be more precise, we put every element of $\mathbf{Q}_{\mathbf{M}_2}$ other than x and y at the top, in the order given by L_3^2, then below them the elements of $\mathbf{Q}_{\mathbf{M}_1}$ other than H, in the order given by L_3^1. Again, the restriction of L_3 to the elements of $\mathbf{Q}_{\mathbf{M}_i}$ is L_3^i, for $i = 1, 2$. Clearly F is the lowest face in L_3.

The fourth extension L_4 is constructed in an analogous manner, putting L_4^1 on top of L_4^2. We claim that the four orders L_j, shown in Figure 3.2, constitute a realizer of $\mathbf{Q}_{\mathbf{M}}$. Clearly they are linear extensions of $\mathbf{Q}_{\mathbf{M}}$: it remains to be shown that every critical pair is reversed.

If (α, β) is a critical pair with α and β both in $\mathbf{Q}_{\mathbf{M}_i}$, for $i = 1$ or 2, then (α, β) is reversed in some L_j^i and so also in L_j.

If v is a vertex in M_1 other than x or y, and J is a face in M_2 other than H, then (v, J) is reversed in L_4. Similarly every critical pair (w, E), where w is a vertex of $\mathbf{M}_2$ and E is a face of $\mathbf{M}_1$, is reversed in L_3.

The only other possible critical pairs are (F, G) and (G, F), and these are reversed in L_4 and L_3, respectively. Therefore $L_1, \dots, L_4$ is an e-realizer of $\mathbf{Q}_{\mathbf{M}}$.

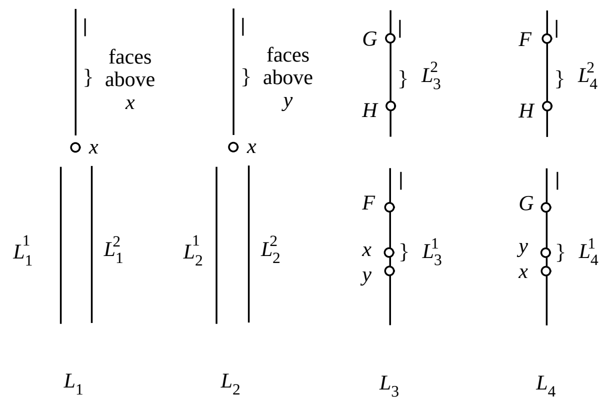

FIG. 3.2. *The new linear extensions* L_j.

(B) *e* is not a critical edge. Let $\{u, v, D, E\}$ be a separating system such that the component $C(e)$ of $\mathbf{R}^2 - \{u, v\} - \text{int}(D) - \text{int}(E)$ containing e is minimal. Let $\mathbf{M}_1$ be the submap determined by the edges in this component together with an edge between u and v separating D and E.

We also form another map $\mathbf{M}_2$ by removing all the edges of $\mathbf{M}_1$ from $\mathbf{M}$ and replacing them with a single edge f between u and v separating D and E. Both $\mathbf{M}_1$ and $\mathbf{M}_2$ have fewer edges than $\mathbf{M}$. See Figure 3.3.

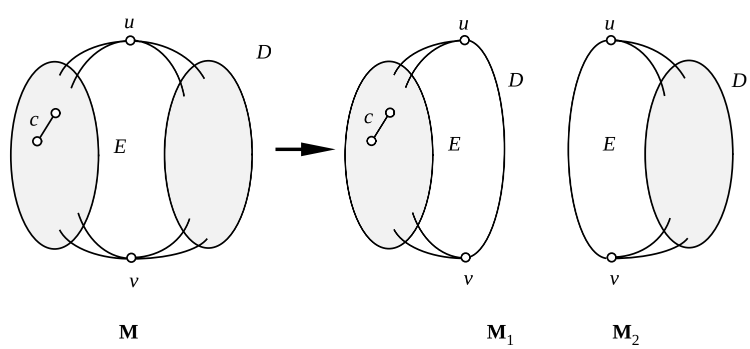

FIG. 3.3. *The maps* $\mathbf{M_1}$ *and* $\mathbf{M_2}$.

Suppose that there is a face H of $\mathbf{M}_1$ other than D and E containing both u and v. Then $\{u, v, D, H\}$ and $\{u, v, E, H\}$ are separating systems in $\mathbf{M}$, and for one of them, say $\{u, v, D, H\}$, the component of e in the complement is a strict subset of $C(e)$. Therefore $uDvH$ is not separating, and so the component of e is the single edge e itself, between u and v. In that case, (e, E) is a critical pair, contradicting the assumption that e is not critical.

Thus D and E are the only faces of $\mathbf{M}_1$ containing both u and v. By duality, we also have that u and v are the only vertices of $\mathbf{M}_1$ on both D and E. In particular, v has a neighbor z on D distinct from u, and there is a face C of $\mathbf{M}_1$ distinct from D and E containing the edge vz. See Figure 3.4.

By a similar argument, we see that neither u nor v is involved in a vertex-vertex critical pair in $\mathbf{Q}_{\mathbf{M}_1}$, and neither D nor E is in a face-face critical pair in $\mathbf{Q}_{\mathbf{M}_1}$.

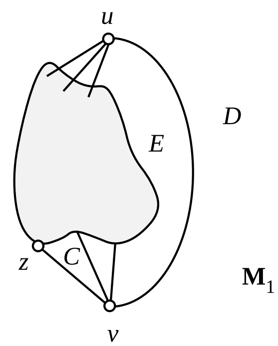

FIG. 3.4. *The vertex z and face C.*

The map $\mathbf{M}_1$ has fewer edges than $\mathbf{M}$, so there is an e-realizer of $\mathbf{Q}_{\mathbf{M}_1}$. In fact, we would like this realizer to have certain extra properties as specified below.

We call a linear extension L of $\mathbf{Q}_{\mathbf{M}_1}$ *u-good* if u is above v and also some face in L. Similarly we call L *v-good* if v is above u and some face in L. The extension L is *D-good* if D is below E and some vertex in L, and L is *E-good* if E is below D and a vertex in L. Note that if $\{L_1, \ldots, L_4\}$ is a realizer of $\mathbf{Q}_{\mathbf{M}_1}$ and $\alpha \in \{u, v, D, E\}$, then one of the L_i is α-good. The next lemma states that rather more is true.

LEMMA 3.4. *There is an e-realizer* (K_u, K_v, K_D, K_E) *of* $\mathbf{Q}_{\mathbf{M}_1}$ *such that, for* $\alpha = u, v, D, E$*, the linear extension* K_α *is* α*-good.*

Note that some of u, v, D, E might coincide with some of x, y, F, G, so the conditions above might preclude (K_u, K_v, K_D, K_E) from being a strong e-realizer.

Proof. Take $\mathcal{R}$ to be an e-realizer of $\mathbf{Q}_{\mathbf{M}_1}$ maximizing the number N of α in the set $\{u, v, D, E\}$ such that there are two α-good linear extensions amongst the linear extensions in $\mathcal{R}$. If $N = 4$, then it is a simple matter to label these linear extensions as K_u, K_v, K_D, K_E in an appropriate manner.

Thus we may assume without loss of generality that only one of the linear extensions is u-good: say L^1 is the only linear extension in $\mathcal{R}$ with u above v and also above some face. In particular, u is above the face C in L^1. Thus the critical pair (z, E) is reversed in some other linear extension, say L^2, of $\mathcal{R}$. Thus L^2 is E-good. A symmetrical argument shows that another linear extension L^3 in $\mathcal{R}$ is D-good. If the last linear extension L^4 of $\mathcal{R}$ is v-good, then we can immediately label the L^i's as (K_u, K_E, K_D, K_v) in that order.

If this is not the case, then L^4 is neither u-good nor v-good, so u and v are both below the lowest face H in L^4. If H does not contain u, then u can be moved to the position immediately above H in L^4: the new set of linear extensions is still an e-realizer, but now both L^1 and L^4 are u-good, and so the value of N is higher for this new set, a contradiction. Similarly if H does not contain v, then v can be moved to the position just above H: this makes L^4 v-good, and so we can label the L^i's as before. Hence we may assume that H contains both u and v and therefore is either D or E—without loss of generality D.

This certainly implies that L^4 is D-good. Now we can apply the same argument as above to L^3 and conclude that D is the lowest face in that order as well. Note that L^2 is necessarily v-good.

It may well be that D is one of F or G, so is forced to be the lowest face in, say, L^3 by the condition that the L^i's form an e-realizer. However, this cannot also be the case in L^4. Also, as in Lemma 3.3, we may assume that all critical pairs involving D are reversed in L^3. Thus D can be moved upward in L^4, and the system is still an e-realizer.

If E is below some vertex in L^4, then putting D at the top of L^4 makes the linear extension E-good, enabling us to label the linear extensions as (K_u, K_v, K_D, K_E). So suppose that E is above all vertices in L^4.

Now put D directly above the second lowest face J in L^4: this keeps L^4 D-good. One of u or v is not on J: place this vertex between D and J. As before, this either increases N or allows a labelling as desired. □

We take an e-realizer (K_u, K_v, K_D, K_E) of $\mathbf{Q}_{\mathbf{M}_1}$ satisfying the conclusions of Lemma 3.4, and a strong f-realizer $\mathcal{S}$ of $\mathbf{Q}_{\mathbf{M}_2}$, and combine them to make an e-realizer of $\mathbf{Q}_{\mathbf{M}}$ as follows.

Consider first the linear extension K_u of $\mathbf{Q}_{\mathbf{M}_1}$, in which u is above v and some face of $\mathbf{M}_1$. We take also that linear extension L_u of $\mathbf{Q}_{\mathbf{M}_2}$ in $\mathcal{S}$, in which u is the top vertex, v the bottom element, and D and E are the top two elements. We combine these to make a linear extension L^u of $\mathbf{Q}_{\mathbf{M}}$ by replacing u in K_u by all of $\mathbf{Q}_{\mathbf{M}_2}$ except for v, D, E, in the order given by L_u. This does indeed give a linear extension of $\mathbf{Q}_{\mathbf{M}}$, and we note also that the top vertex and bottom face in L^u are the same as in K_u. See Figure 3.5.

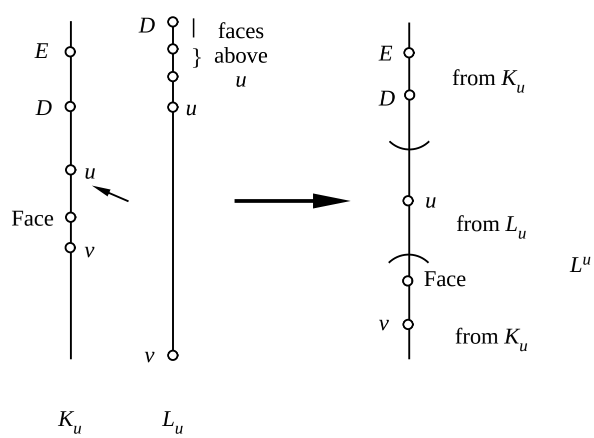

FIG. 3.5. *The new linear extensions.*

We repeat with the other linear extensions to obtain four linear extensions L^u, L^v, L^D, L^E of $\mathbf{Q}_{\mathbf{M}}$. It remains to be shown that these form a realizer. Notice that if, for instance, u is above a face H in K_α, then every vertex in $\mathbf{M}_2$, other than perhaps v, comes above H in L^α.

We consider each possible type of critical pair in turn, checking it is reversed by one of the four linear extensions.

We start with critical pairs involving u. For H a face in $\mathbf{M}_1$ not including u, (u, H) is reversed in L^α whenever it is reversed in K_α. For β an element of $\mathbf{M}_2$ with (u, β) a critical pair, (u, β) is reversed in L_u, and hence also in L^u. Similarly all critical pairs involving v, D, or E are catered to.

Let z be a vertex of $\mathbf{M}_1$ other than u and v. Without loss of generality z is not on the face E, so the pair (z, E) is reversed in some K_α. Hence all the faces of $\mathbf{M}_2$ come below z in L^α, so all critical pairs of the form (z, H) for H a face of $\mathbf{M}_2$ are reversed in L^α. By duality, all pairs of the form (w, J) for w a vertex of $\mathbf{M}_2$ and J a face in $\mathbf{M}_1$ are also reversed.

Finally, if β and γ are elements of the same $\mathbf{M}_i$, then if (β, γ) is a critical pair then it is reversed in some K_α or L_α, and hence is reversed in the corresponding L^α.

Thus every critical pair is reversed by some L^α, and so the family (L^u, L^v, L^D, L^E) constitutes a realizer. Since the top and bottom elements are the same in L^α as in K_α, this is an e-realizer.

In both cases, we have constructed an e-realizer for our poset $\mathbf{Q_M}$. Thus, by induction, $\mathbf{Q_M}$ has an e-realizer for every 2-connected map $\mathbf{M}$ and edge e. □

4. Concluding remarks. It is proved in Reuter [3], and in [1], that, for every 3-connected map $\mathbf{M}$, $\dim(\mathbf{Q_M}) \geq 4$, and therefore $\dim(\mathbf{P_M}) = \dim(\mathbf{Q_M}) = 4$. Obviously this is not true if the 3-connectedness condition is removed, and we are left with the questions of characterizing the planar maps $\mathbf{M}$ with $\dim(\mathbf{P_M})$ or $\dim(\mathbf{Q_M})$ equal to 3 (or 2). We offer a few remarks on some of these problems.

Let us first ask which maps $\mathbf{M}$ have $\dim(\mathbf{P_M})$ equal to 2. Note that, if $\mathbf{M}$ contains any cycle with at least 3 vertices, then $\dim(\mathbf{P_M}) \geq 3$, since the subposet of $\mathbf{P_M}$ induced by the vertices and edges of the cycle is a crown. If $\mathbf{M}$ contains any edges with multiplicity at least 3, they give rise to a cycle in the dual, so again $\mathbf{P_M}$ has dimension at least 3. Similarly, if any vertex (face) of $\mathbf{M}$ has three distinct neighbors, then $\dim(\mathbf{P_M}) \geq 3$. Hence, if $\dim(\mathbf{P_M}) = 2$, then each component of the underlying graph of M is a path, possibly with loops and/or double edges. Similar considerations lead to the conclusions that only the final edges of paths can be double edges, that all loops separate one endvertex of the path from the other, and that, if X and Y are two components of the graph, then an endvertex of X must share a face with an endvertex of Y. These restrictions give us a complete characterization of maps $\mathbf{M}$ with $\dim(\mathbf{P_M}) = 2$: a typical such map is shown in Figure 4.1.

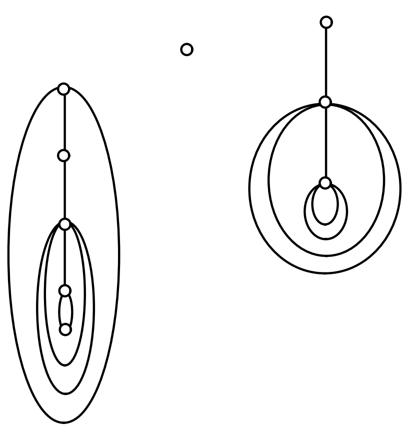

FIG. 4.1. *A map* $\mathbf{M}$ *with* $\mathbf{P}_M = 2$.

As far as we can tell, none of the other three problems suggested at the beginning of this section has as neat a solution. Maybe the right question is, are there polynomial algorithms to determine whether $\dim(\mathbf{P_M})$ or $\dim(\mathbf{Q_M})$ is equal to 3? It is known that this problem for a general partial order is NP-complete, but there is a polynomial algorithm to determine whether a partial order has dimension 2.

Another related line of inquiry is to ask which maps $\mathbf{M}$ have $\mathbf{Q_M}$ 4-irreducible. We know from [1] that all 3-connected maps have this property, and it is tempting to conjecture the converse: if $\mathbf{Q_M}$ is 4-irreducible, then $\mathbf{M}$ is 3-connected. However, the example in Figure 4.2 shows that this is false.

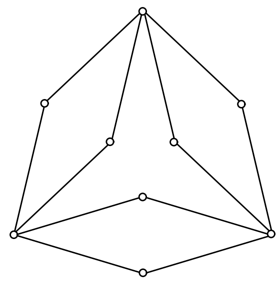

FIG. 4.2. *A non-3-connected map* $\mathbf{M}$ *with* $\mathbf{Q}_M$*4-irreducible.*

Again, we suspect that there is no particularly neat characterization, and the complexity version of the problem may be more fruitful.

Finally, it is natural to ask how the results of [1] and this paper extend to other surfaces. If $\mathbf{M}$ is a map drawn on a surface of genus k, then there are some bounds $f(k), g(k)$ for $\dim(\mathbf{P_M})$ and $\dim(\mathbf{Q_M})$. What are the best possible bounds? Are they the same in both cases? We tentatively venture the suggestion that $\dim(\mathbf{P_M})$ and $\dim(\mathbf{Q_M})$ are still bounded above by 4 when $\mathbf{M}$ is a map drawn on the torus.

REFERENCES

[1] G. R. BRIGHTWELL AND W. T. TROTTER, *The order dimension of convex polytopes*, SIAM J. Discrete Math., 6 (1993), pp. 230–245.

[2] D. KELLY AND W. T. TROTTER, *Dimension theory for ordered sets*, in Proceedings of the Symposium on Ordered Sets, I. Rival et al., eds., Reidel, Boston, MA, 1982, pp. 171–212.

[3] K. REUTER, *On the Order Dimension of Convex Polytopes*, preprint.

[4] W. SCHNYDER, *Planar graphs and poset dimension*, Order, 15 (1989), pp. 323–343.

[5] W. T. TROTTER, *Progress and new directions in dimension theory for finite partially ordered sets*, in Extremal Problems for Finite Sets, P. Frankl et al., eds., Bolyai Soc. Math. Studies 3, 1991, pp. 457–477.

[6] W. T. TROTTER, *Combinatorics and Partially Ordered Sets: Dimension Theory*, The Johns Hopkins University Press, Baltimore MD, 1992.