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# Dimension and height for posets with planar cover graphs

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ABSTRACT

We show that for each integer $h \geq 2$, there exists a least positive integer $c_h$ so that if $P$ is a poset having a planar cover graph and the height of $P$ is $h$, then the dimension of $P$ is at most $c_h$. Trivially, $c_1 = 2$. Also, Felsner, Li and Trotter showed that $c_2$ exists and is 4, but their proof techniques do not seem to apply when $h \geq 3$. We focus on establishing the existence of $c_h$, although we suspect that the upper bound provided by our proof is far from best possible. From below, a construction of Kelly is easily modified to show that $c_h$ must be at least $h + 2$.



## 1. Introduction

We assume the reader is familiar with basic combinatorial concepts for finite partially ordered sets: cover graphs, comparability graphs, order diagrams, maximal and minimal elements, chains, antichains, height and width. We also assume some familiarity with the concept of dimension for posets and the role of critical pairs and alternating cycles in determining dimension. Readers who would like additional background material may find it helpful to consult [16,17].

In this paper, we focus on combinatorial problems associated with order diagrams and cover graphs. In some sense, it is easy to characterize graphs that are cover graphs, as we have the following self-evident proposition: a graph $G$ is a cover graph if and only if the edges of $G$ can be oriented so that there are no oriented paths $x_1, x_2, x_3, \dots, x_n$ where $n \geq 3$ and $x_1x_n$ is an edge in $G$. Nevertheless, it is quite difficult to devise an algorithm for implementing this test; in fact, Nešetřil and Rödl [14] and Brightwell [4] have shown that answering whether a graph is a cover graph is NP-complete.

A poset $P$ is said to be *planar* if it has an order diagram without edge crossings. If a poset is planar, then its cover graph is planar, but the converse need not be true. Although height two posets with planar cover graphs also have planar diagrams [13,7], for all $h \geq 3$, there exist height $h$ non-planar posets with planar cover graphs. Also, while there are very fast algorithms for testing graph planarity,

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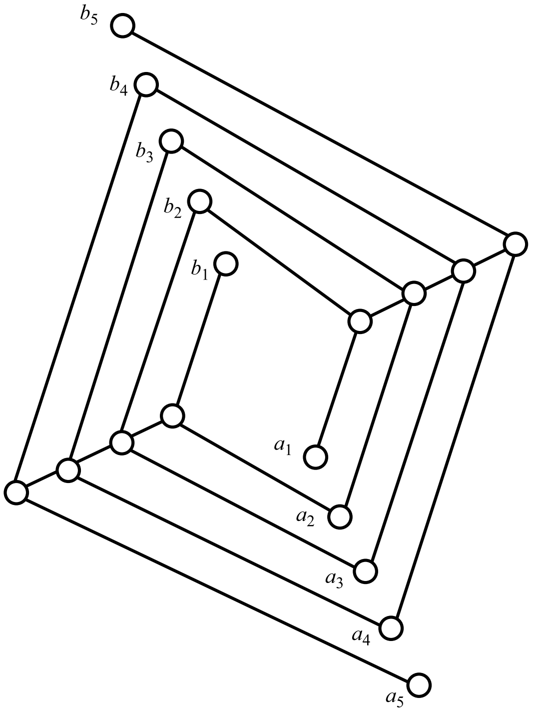


**Fig. 1.** Kelly's construction.

with running time linear in the number of edges [11], Garg and Tamassia [10] showed that it is NP-complete to answer whether a poset is planar.

Recall that the *dimension* of a poset $P$, denoted $\dim(P)$, is the least positive integer $t$ for which there are linear orders $L_1, L_2, \ldots, L_t$ on the ground set of $P$ so that $P = L_1 \cap L_2 \cap \cdots \cap L_t$. The following comprehensive theorem summarizes previously known results connecting dimension and planarity for posets. These results are proved in [1,18,12], respectively.

**Theorem 1.1.** *Let P be a finite poset.*

(1) *If P has a zero and a one, then P is planar if and only if P is a* 2*-dimensional lattice.*
(2) *If P has a zero (or a one), then the dimension of P is at most* 3.
(3) *There exist planar posets with arbitrarily large dimension.*

For $n \geq 2$, the *standard example* $S_n$ is a height two poset with minimal elements $a_1, a_2, \ldots, a_n$, maximal elements $b_1, b_2, \ldots, b_n$, with $a_i < b_j$ in $S_n$ if and only if $i \neq j$. It is well-known that $\dim(S_n) = n$. Furthermore, $S_n$ is irreducible when $n \geq 3$, i.e., the removal of any point from $S_n$ decreases the dimension to $n-1$. For $n \leq 4$, $S_n$ is planar, so there exist planar posets of dimension 4. For $n \geq 5$, even the cover graph of $S_n$ is non-planar. However, the proof given by Kelly [12] demonstrating that there are planar posets with arbitrarily large dimension actually shows that for each $n \geq 5$, the standard example $S_n$ is a *subposet* of a planar poset. We illustrate Kelly's construction in Fig. 1 for the specific value $n = 5$, noting that the construction is easily generalized when $n \geq 6$.

## 2. Planar graphs and dimension

Recall that the *vertex–edge* poset of a graph $G$ is the height two poset $P_G$ having the vertices of $G$ as minimal elements and the edges of $G$ as maximal elements, with $x < e$ in $P_G$ if and only if $x$ is an end of $e$ in $G$. In 1989 Schnyder [15] proved the following now classic result.

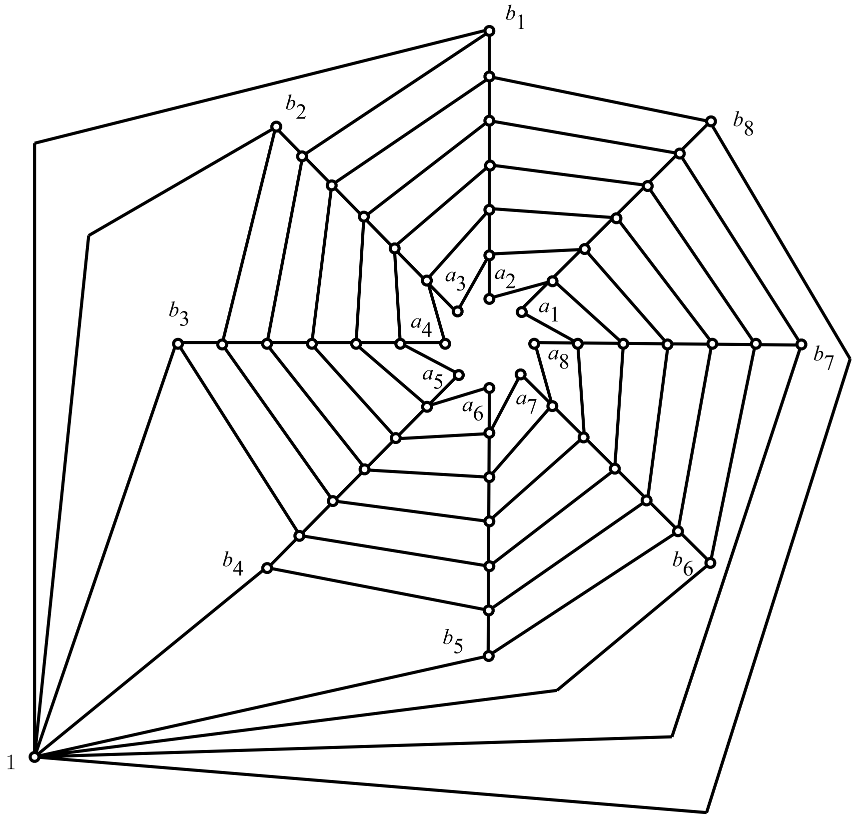


**Fig. 2.** A poset with a one and a planar cover graph.

**Theorem 2.1.** *Let G be a graph and let $P_G$ be the vertex–edge poset of G. Then G is planar if and only if* $\dim(P_G) \leq 3$.

The machinery developed by Schnyder in his proof of Theorem 2.1 has led to deep insights in other areas of mathematics, such as graph drawing (e.g. see [8]). However, Barrera-Cruz and Haxell [2] have recently provided a shorter proof avoiding Schnyder's machinery. In [5,6], Brightwell and Trotter extended Schnyder's theorem, but it is the second of these results which is more central to this paper.

**Theorem 2.2.** *Let P be the vertex–edge–face poset of a planar multi-graph drawn without edge crossings in the plane. Then* $\dim(P) \leq 4$.

## 3. Posets having a planar cover graph

We show in Fig. 2 a planar cover graph of a poset $P$ that (1) has a one and (2) contains the standard example $S_8$. Again, this drawing is just one instance of an infinite family and shows that there is no analogue of the second part of Theorem 1.1 for cover graphs.

A poset of height 1 is an antichain, and non-trivial antichains have dimension 2. For height 2 posets, we have the following theorem proved by Felsner, Li and Trotter [9].

**Theorem 3.1.** *Let P be a poset of height 2. If the cover graph of P is planar, then* $\dim(P) \leq 4$.

The standard example $S_4$ shows that the inequality in Theorem 3.1 is best possible. Motivated by this theorem and the fact that the posets in Kelly's construction have large height, Felsner, Li and

Trotter conjectured in [9] that a poset with a planar cover graph has dimension which can be bounded in terms of its height. The primary result of this paper will verify this conjecture.

**Theorem 3.2.** *For every $h \geq 1$, there exists a least positive integer $c_h$ so that if $P$ is a poset of height $h$ and the cover graph of $P$ is planar, then* $\dim(P) \leq c_h$.

We note that the proof of Theorem 3.1 proceeds by showing that $P$ is isomorphic to the vertex–face poset of a planar map, so that the upper bound from Theorem 2.2 may be applied. Independent of this machinery, we know of no entirely simple[1] argument to show that the dimension of a poset of height 2 having a planar cover graph is bounded—even by a very large constant. Furthermore, we do not see how the techniques developed in [9] can be extended to the case $h \geq 3$.

## 4. Proof of the main theorem

Our proof will utilize the following basic concepts concerning dimension. A set of incomparable pairs in a poset $P$ is *reversible* if there is a linear extension $L$ of $P$ with $x > y$ in $L$ for every $(x, y)$ in the set. Also, a set $\{(x_i, y_i) : 1 \leq i \leq k\}$ of incomparable pairs is called a *strict alternating cycle* (of length $k$) when $x_i \leq y_j$ in $P$ if and only if $j = i + 1$ (cyclically). A set of incomparable pairs is reversible if and only if it does not contain a strict alternating cycle. An incomparable pair $(x, y)$ is called a *critical pair* when $u < x$ implies $u < y$ in $P$ and $w > y$ implies $w > x$. The dimension of a poset is one if and only if it is a linear order, i.e., there are no incomparable pairs. When $P$ is not a linear order, the dimension of $P$ is then the least positive integer $t$ for which the set $\text{Crit}(P)$ of all critical pairs of $P$ can be partitioned into $t$ reversible subsets.

Now on with the proof. We assume that $P$ is a poset of height $h \geq 3$ and that $P$ has a planar cover graph $G$. Clearly, we may assume that $G$ is connected. To show that $\dim(P)$ is bounded in terms of $h$, it is enough to show that we may partition the set $\text{Crit}^*(P)$ of incomparable min–max pairs[2] into a small number of reversible sets, where small means bounded as a function of $h$.

To accomplish this task, we will provide for each critical pair $(a, b)$ from $\text{Crit}^*(P)$ a *signature*. The reader should think of a signature as a vector of parameters, although we do not require that these vectors have a common length, nor do we require that the $i$th coordinate of every vector represents the same parameter. However, we do require (1) the number of parameters in the signature is bounded as a function of $h$, and (2) the number of distinct values that can be taken by any given coordinate in the signature is bounded as a function of $h$. As a consequence of these two conditions, the number of distinct signatures is also bounded as a function of $h$. Finally, we will show that any set of critical pairs with identical signatures is reversible.

We first handle a special case—although as we will see, this case is actually the heart of the problem. We then return to the general case in Section 7.

*Special case.* There is an $a_0 \in \min(P)$ such that $a_0 < b$ in $P$ for all $b \in \max(P)$.

Consider a plane drawing without edge crossings of $G$ with the vertex $a_0$ on the infinite face. We consider the edges of $G$ oriented from $u$ to $v$ when $u < v$ in $P$. An oriented path $Q = (u_0, u_1, \ldots, u_t)$ from $u = u_0$ to $v = u_t$ witnesses that $u_0 < u_t$ in the poset $P$. Frequently, we will refer to such a path as $Q(u, v)$ to emphasize that $Q$ starts at $u$ and ends at $v$. When $0 \leq i \leq j \leq t$, $Q(u_i, u_j)$ will denote the portion of $Q$ starting with $u_i$ and ending with $u_j$. In this paper, all oriented paths will be denoted with the letters $Q$, $L$ and $R$.

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1 Felsner, Li and Trotter also showed that the dimension of a poset of height two can be bounded as a function of the acyclic chromatic number of the cover graph. Since a planar graph has acyclic chromatic number at most five [3], this yields a bound on the dimension of the poset. However, using the techniques of [9], the resulting bound is 65, and while the argument can no doubt be tightened, it is unlikely to yield the correct answer, which is four. This technique fails for $h \geq 3$, as demonstrated by the poset obtained by subdividing each comparability of $S_n$ (i.e. for each pair $(i, j)$ with $i \neq j$, add $c_{ij}$ and comparabilities $a_i < c_{ij} < b_j$). Coloring all of the $a$'s with color 1, the $b$'s with color 2, and $c$'s with color 3, we find that the acyclic chromatic number of the cover graph is at most 3, whereas the dimension of the poset is $n$.

2 In general, it is necessary to reverse *all* critical pairs, but for posets with planar cover graphs, it is easy to add new points so that reversing pairs in $\text{Crit}^*(P)$ is enough.

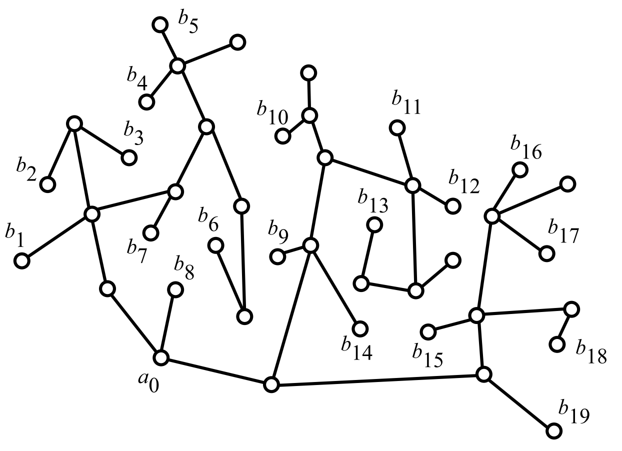


**Fig. 3.** The oriented tree $T$.

However, we will also discuss paths, cycles and walks in $G$ in the general sense, i.e., without any concern for the orientation on the edges. In particular, we will use the letters $T$ and $S$ to denote trees, where the vertex sets of such trees will be a proper subset of the vertex set of $G$. When $u$ and $v$ are distinct vertices in the tree $T$, we use $T(u, v)$ to denote the unique path in $T$ starting with $u$ and ending with $v$. In general, $T(u, v)$ is not an oriented path.

For convenience, we let $A$ denote the set $\min(P) - a_0$ and we let $B = \max(P)$. Let $T$ be an oriented tree so that:

(1) $T$ is a subgraph of $G$;
(2) $a_0$ is the root of $T$;
(3) all other vertices in $T$ distinct from $a_0$ are on paths oriented away from $a_0$; and
(4) the elements of $B$ are leaves of $T$ (although perhaps there are leaves of $T$ that are not in $B$).

Using clockwise orientation to establish precedence, we perform a depth first search of $T$ and this results in a linear order on the vertices of $T$ with the root $a_0$ as the least element. If an element $x$ is less than another element $y$ in this linear order then we write $x <_T y$. We illustrate in Fig. 3 what the resulting order on the elements of $B$ would be.

One word of caution about Fig. 3. In this figure, we mean that $b_i <_T b_j$ whenever $1 \le i < j \le 19$. However, in the discussions to follow, we will discuss elements of $B$ with subscripts which do not necessarily reflect their order in the tree $T$. In particular, when we say that $b_1$ and $b_2$ are elements of $B$, we are not implying that $b_1 <_T b_2$.

As a second example, we return to Fig. 2 and relabel the point which was previously a one to be a minimal element $a_0$ which is less than each maximal element. Now we have a poset $P$ satisfying the properties we are assuming in this special case, and we have a suitable drawing with the vertex $a_0$ on the infinite face. It follows in this example that the oriented tree $T$ is just a star, and the resulting linear order is $a_0 <_T b_1 <_T b_2 <_T \cdots <_T b_8$.

When $u$ and $v$ are distinct vertices in $T$, we let $|T(u, v)|$ count the number of vertices in the path $T(u, v)$. For brevity, when $u = a_0$, we write $T(v)$ rather than $T(a_0, v)$. Furthermore, we refer to the quantity $h(v) = |T(v)|$ as the *height* of $v$ in $T$.

Now let $a \in A$. Set $\text{Spec}(a) = \{s \in T : a < s \text{ in } P \text{ and } a \parallel u \text{ for all } u \in T(s) \text{ with } u \neq s\}$. We say the elements of $\text{Spec}(a)$ are the *special points of* $a$. When $s$ and $s'$ are distinct special points in $\text{Spec}(a)$, it may happen that $s < s'$ in $P$. However, $s$ cannot be on the path $T(s')$. The following elementary statement is so important to our argument that it deserves to be listed as a proposition.

**Proposition 4.1.** *If $a \in A$, $u \in T$ and $a < u$ in $P$, then there is some $s \in \text{Spec}(a)$ so that $s$ is on the path $T(u)$.*

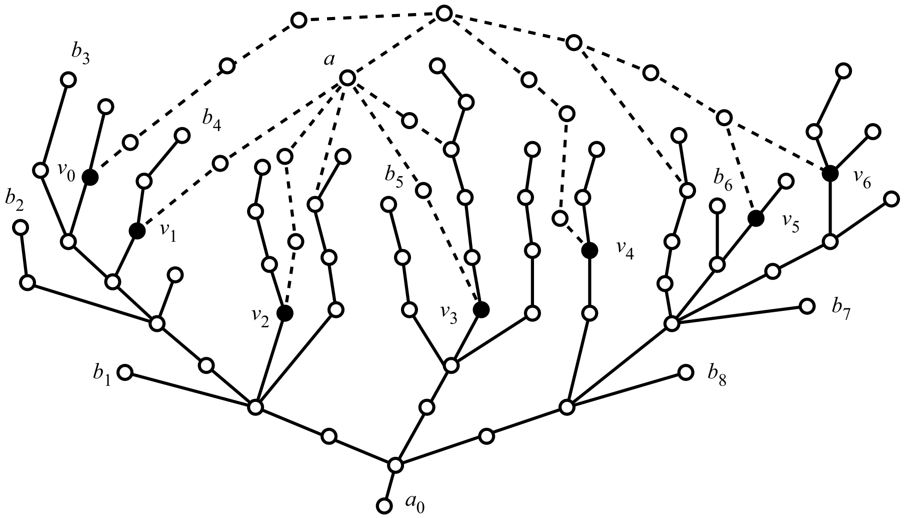


**Fig. 4.** Unimodal sequence.

## 5. Unimodal sequences

For each $a \in A$, we define two sequences. One is a unimodal sequence $(h_0(a), h_1(a), \ldots, h_r(a))$ of positive integers. In particular, there is some integer $i \geq 0$ for which

$$h \geq h_0(a) > h_1(a) > \cdots > h_i(a) \leq h_{i+1}(a) < h_{i+2}(a) < \cdots < h_r(a) \leq h.$$

This sequence can be trivial and consist of a single integer. We also have a sequence $(v_0(a), v_1(a), \ldots, v_r(a))$ of distinct points from Spec$(a)$. These two sequences are uniquely defined by the following requirements.

(1) $v_0(a) <_T v_1(a) <_T \cdots <_T v_r(a)$.
(2) For each $j = 0, 1, \ldots, r$, $h_j(a) = h(v_j(a))$.
(3) $h_i(a) = \min\{h(s) : s \in \text{Spec}(a)\}$.
(4) If $0 \leq j < r$, $s \in \text{Spec}(a)$ and $v_j(a) <_T s <_T v_{j+1}(a)$, then $h(s) \geq \max\{h_j(a), h_{j+1}(s)\}$.

The sequence $(h_0(a), h_1(a), \ldots, h_r(a))$ is called the *unimodal sequence* of the element $a$. The corresponding sequence of points $(v_0(a), v_1(a), \ldots, v_r(a))$ is called the *unimodal sequence of special points of* $a$. Note that $v_0(a)$ is the $<_T$-least element of Spec$(a)$, while $v_r(a)$ is the $<_T$-largest element of Spec$(a)$. We illustrate this definition in Fig. 4 where we show a minimal element $a$ with unimodal sequence $(10, 9, 6, 6, 7, 8, 9)$. In this figure, the solid lines represent edges in the tree $T$ while the dashed lines represent edges oriented away from $a$.

Clearly, the number of unimodal sequences is bounded as a function of $h$, and the unimodal sequence of a minimal element $a$ will be added to the signature of all critical pairs $(a, b)$ with $a$ as first coordinate. On the other hand, the unimodal sequence of special points of $a$ will *not* be part of the signature, as the number of such sequences is not bounded in terms of $h$. Since we will be considering sets of critical pairs with the same signature, we will often drop the reference to $a$ and denote the terms of the unimodal sequence as $(h_0, h_1, \ldots, h_r)$.

### 5.1. Safe and dangerous critical pairs

Let $(a, b) \in \text{Crit}^*(P)$. We say that $(a, b)$ is *left-safe* if $b <_T s$ for every $s \in \text{Spec}(a)$. Similarly, we say that $(a, b)$ is *right-safe* if $s <_T b$ for every $s \in \text{Spec}(a)$. In Fig. 4, $(a, b_1)$, $(a, b_2)$ and $(a, b_3)$ are left-safe while $(a, b_7)$ and $(a, b_8)$ are right-safe.

**Proposition 5.1.** *The following two subsets of* Crit$^*(P)$ *are reversible:*

(1) $\{(a, b) \in \text{Crit}^*(P) : b$ *is left-safe for* $a\}$.
(2) $\{(a, b) \in \text{Crit}^*(P) : b$ *is right-safe for* $a\}$.

**Proof.** Suppose that the proposition fails for the first set. Choose an integer $k \geq 2$ and a strict alternating cycle $\{(a_i, b_i) : 1 \leq i \leq k\}$ with $b_i$ left-safe for $a_i$ for each $i = 1, 2, \ldots, k$. For each $i$, let $s_i$ denote the $<_T$-least element of Spec$(a_i)$. Also, for each $i = 1, 2, \ldots, k$, since $a_i \leq b_{i+1}$, there is a point $t_i \in \text{Spec}(a_i)$ with $a_i < t_i \leq b_{i+1}$ in $P$ and $t_i$ on $T(b_{i+1})$. But this implies $t_i \leq_T b_{i+1} <_T s_{i+1} \leq_T t_{i+1}$, which cannot hold cyclically. The proof for the second set is the same. □

So for the remainder of the proof, we consider only critical pairs $(a, b)$ in Crit$^*(P)$ which are neither left-safe or right-safe. We call these pairs *dangerous*. In Fig. 4, $(a, b_4)$, $(a, b_5)$ and $(a, b_6)$ are dangerous. When $(a, b)$ is a dangerous critical pair, there is a unique integer $j$ so that $v_j(a) <_T b <_T v_{j+1}(a)$. The integer $j$ is called the *location* of $b$. In Fig. 4, the location of $b_4$ is 0, the location of $b_5$ is 2 and the location of $b_6$ is 4.

In view of these remarks, we fix a positive integer $r$, a sequence $\sigma = (h_0, h_1, \ldots, h_r)$ and an integer $j$ with $0 \leq j < r$. We then consider the subfamily Crit$^*(P, \sigma, j)$ of all critical pairs $(a, b)$ from Crit$^*(P)$ where (1) $\sigma$ is the unimodal sequence of $a$, and (2) the location of $b$ is $j$. We assume that Crit$^*(P, \sigma, j)$ is non-empty, and we will focus on strategies for partitioning Crit$^*(P, \sigma, j)$ into a small number (bounded as a function of $h$) of reversible subfamilies.

Since the poset $P$, the unimodal sequence $\sigma$ and the integer $j$ are fixed, we will simplify notation and just write Crit$^*$ rather than Crit$^*(P, \sigma, j)$. As the argument proceeds and we identify additional parameters to be added to the signature, we will always assume that Crit$^*$ denotes a subfamily on which are relevant parameters are constant.

### *5.2. Minimal regions*

Let $(a, b)$ be a critical pair from Crit$^*$. Set $x = v_j(a)$ and $y = v_{j+1}(a)$. Then let $L = L(a, x)$ and $R = R(a, y)$ be oriented paths in $G$, and let $m = m(L, R)$ be the last point of $L$ which is also a point on $R$. The paths $L(m, x)$ and $R(m, y)$ together with the path $T(x, y)$ in $T$ form the boundary of a region in the plane. The family of all regions formed in this manner is partially ordered by inclusion, and we choose paths $L$ and $R$ so that the region is a minimal element in this partial order. This minimal region is denoted $\mathcal{R}_j(a)$, and the last point $L$ and $R$ have in common is denoted $m_j(a)$. The boundary of $\mathcal{R}_j(a)$ is a simple closed curve formed by the three defining paths $L(m, x)$, $R(m, y)$ and $T(x, y)$. Note that $a_0$ does not belong to the interior of $\mathcal{R}_j(a)$, but of course, it may belong to its boundary. Also, it may happen that $a = m_j(a)$.

We add to the signature of $(a, b)$ the quantities $|L(m, x)|$ and $|R(m, y)|$. Letting $p$ be the last common point of $T(x)$ and $T(y)$, we also record $h(p)$ in the signature of $(a, b)$. Of course, we do not include mention of the specific points $m, x, y$ or $p$ in the signature of $(a, b)$.

The next proposition is self-evident, but it may help the reader to see why we have introduced regions.

**Proposition 5.2.** *Let* $(a, b)$ *be a dangerous critical pair from* Crit$^*$. *Then* $b$ *is in the interior of the region* $\mathcal{R}_j(a)$.

The following propositions, which use the notation of the preceding discussion, are stated without proof, as they admit quite elementary proofs; however, they are key to future arguments.

**Proposition 5.3.** *There is no non-trivial oriented path* $Q$ *in* $G$ *starting at a point from* $L(m, x) \cup R(m, y)$ *and ending at a point from the boundary of* $\mathcal{R}_j(a)$ *with all edges of* $Q$ *in the interior of* $\mathcal{R}_j(a)$.

We point out that there is a basic inconsistency in our notation, but it is so minor that we feel no confusion can arise. When $a_1$ and $a_2$ are distinct, but there is some point $x$ such that $x = v_j(a_1) = v_j(a_2)$, we refer to the paths $L(m_1, x)$ and $L(m_2, y)$. This might be construed to mean that $L(m_1, x)$ and $L(m_2, y)$ are both part of a single path, but in fact, they are just oriented paths with the same end point.

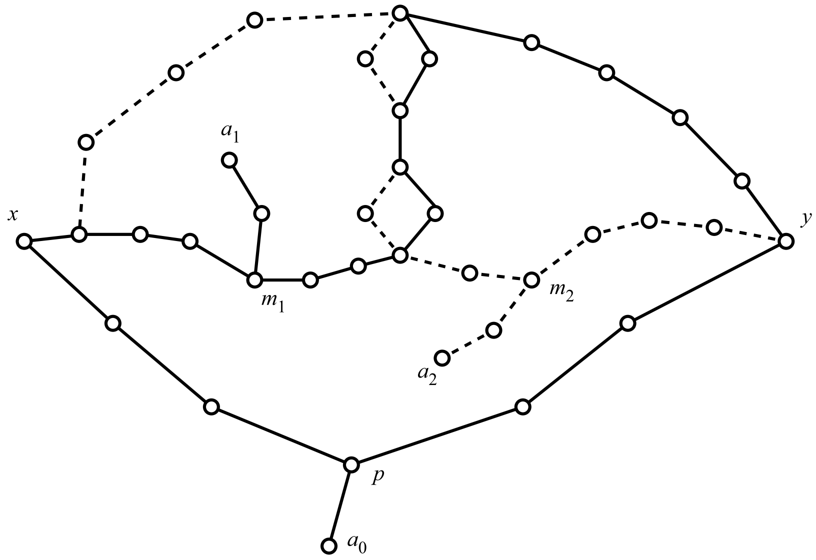


**Fig. 5.** Incomparable regions.

Since the location $j$ is fixed, when $(a, b)$ is a critical pair from $\text{Crit}^*$, we will again abuse notation slightly and refer to the region $\mathcal{R}_j(a)$ as $\mathcal{R}(a)$. Also, we just write $m(a)$ rather than $m_j(a)$. With this understanding, when $\{(a_i, b_i) : 1 \le i \le t\}$ is a family of critical pairs from $\text{Crit}^*$, we may use the shorthand notation: $x_i = v_j(a_i)$, $y_i = v_{j+1}(a_i)$, $m_i = m_j(a_i)$ and $\mathcal{R}_i = \mathcal{R}_j(a_i)$.

**Proposition 5.4.** *Let $(a_1, b_1)$ and $(a_2, b_2)$ be critical pairs from* $\text{Crit}^*$ *with $a_1 \ne a_2$. Suppose that there is a point $x$ with $x = x_1 = x_2$. If $u$ is a vertex common to both $L(m_1, x)$ and $L(m_2, x)$, then the portion of these two paths starting at $u$ and ending at $x$ is identical.*

There is an obvious dual form of the preceding proposition for paths $R(m_1, y)$ and $R(m_2, y)$ which we will not state formally. However, the following consequence follows immediately.

**Proposition 5.5.** *Let $(a_1, b_1)$ and $(a_2, b_2)$ be critical pairs from* $\text{Crit}^*$ *with $a_1 \ne a_2$. Suppose there are distinct points $x$ and $y$ so that $x = x_1 = x_2$ and $y = y_1 = y_2$. Then the following statements hold.*

(1) *If $m_1$ is not in the interior of $\mathcal{R}_2$, then $\mathcal{R}_2$ is a proper subset of $\mathcal{R}_1$.*
(2) *If $m_2$ is not in the interior of $\mathcal{R}_1$, then $\mathcal{R}_1$ is a proper subset of $\mathcal{R}_2$.*

We continue with the notation of Proposition 5.5 and assume that we have regions $\mathcal{R}_1$ and $\mathcal{R}_2$, neither of which is a subset of the other. In this case, it is easy to see that exactly one of the following two statements is valid.

(1) $R(m_1, y)$ intersects $L(m_2, x)$.
(2) $R(m_2, y)$ intersects $L(m_1, x)$.

If the first statement holds, we say $a_1$ is *left* of $a_2$. Otherwise, $a_2$ is left of $a_1$. Clearly, when $x$ and $y$ remain fixed, this concept is transitive, i.e., if $a_1$ is left of $a_2$ and $a_2$ is left of $a_3$, then $a_1$ is left of $a_3$.

We illustrate this situation in Fig. 5, where the paths for $a_1$ are shown with solid lines while the paths for $a_2$ are dashed. Here, $a_1$ is left of $a_2$. Note that the paths $R(m_1, y)$ and $L(m_2, x)$ may interweave and may even share edges. In this figure, shared edges are solid. Also, we show $a_1$ as belonging to the exterior of $\mathcal{R}_1$ while $a_2$ is in the interior of $\mathcal{R}_2$. Any combination is possible.

Let $\{(a_i, b_i) : 1 \le i \le t\}$ be a sequence of critical pairs from $\text{Crit}^*$. We say that $(a_1, a_2, \dots, a_t)$ is a *left-to-right sequence* when (1) there are points $x$ and $y$ so that $x = x_i$ and $y = y_j$ for each $i = 1, 2, \dots, t$, and (2) for each $i = 1, 2, \dots, t - 1$, $\mathcal{R}_i$ and $\mathcal{R}_{i+1}$ are incomparable regions, with $a_i$ to the left of $a_{i+1}$. Of course, this definition requires that the elements of $\{a_i : 1 \le i \le t\}$ are distinct.

**Lemma 5.6.** *For fixed special points x and y, the length of a left-to-right sequence* $(a_1, a_2, \ldots, a_t)$ *is bounded as a function of h.*

**Proof.** We apply Ramsey theory, coloring the 2-element subsets of $\{1, 2, \ldots, t\}$ with $h^2$ colors. For each 2-element set $\{i_1, i_2\}$ with $1 \le i_1 < i_2 \le t$, consider the first point $q$ common to $R(m_{i_1}, y)$ and $L(m_{i_2}, x)$. Note that $q$ is distinct from $m_{i_1}$ and $m_{i_2}$. Assign to the pair $\{i_1, i_2\}$ the color $(c, d)$ where $c = |R(m_{i_1}, q)|$ and $d = |L(m_{i_2}, q)|$. If $t$ is sufficiently large, then there is a three element monochromatic set $\{i_i < i_2 < i_3\}$. This implies that there is some point $q$ distinct from $m_1, m_2$ and $m_3$ where $R(m_{i_1}, y)$, $R(m_{i_2}, y)$, $L(m_{i_2}, x)$ and $L(m_{i_3}, x)$ all meet. This is impossible since $L(m_{i_2}, x)$ and $R(m_{i_2}, y)$ have no common point except for $m_{i_2}$. □

In view of this lemma, it is natural to add to the signature of a critical pair $(a, b)$ the maximum value $t$ for which there is a left-to-right sequence $(a_1, a_2, \ldots, a_t)$ of elements with $a = a_1$. It follows that if $(a_1, b_1)$ and $(a_2, b_2)$ are distinct elements of $\text{Crit}^*$ with the same signature, and there are elements $x$ and $y$ so that $x = x_i$ and $y = y_j$ for $i = 1, 2$, then one of $\mathcal{R}_1$ and $\mathcal{R}_2$ is a subset of the other. Otherwise, we may assume that $a_1$ is to the left of $a_2$, which would imply that there is a longer left-to-right sequence starting with $a_1$ than there is starting with $a_2$.

### 5.3. Identical regions

Now we consider a region $\mathcal{R}$ and the subfamily $\text{Crit}^*(\mathcal{R})$ consisting of all pairs $(a, b)$ from $\text{Crit}^*$ with (1) $\mathcal{R} = \mathcal{R}(a)$ and (2) $a$ in the interior of $\mathcal{R}$. For this subfamily, there are points $m$, $x$ and $y$ so that the boundary of $\mathcal{R}$ is formed by the paths $L(m, x)$, $R(m, y)$ and $T(x, y)$. Consider the set $S$ consisting of $m$ and the points in the interior of $\mathcal{R}$ which are less than $m$ in $P$. We choose appropriate edges from $G$ so that $S$ can be considered as a tree with $m$ as root. In this tree, all other points are on paths oriented towards $m$. Using clockwise orientation, we carry out a depth-first search of $S$ to determine a linear order $<_S$ on the vertices in $S$. As before, $m$ is the least element. In discussions to follow, we will use the same conventions for paths in the tree $S$ that we have used for paths in the tree $T$.

An ordered pair $((a_1, b_1), (a_2, b_2))$ of critical pairs from $\text{Crit}^*(\mathcal{R})$ is called a *reversing pair* if

(1) $a_1 < b_2$ and $a_2 < b_1$ in $P$.
(2) $b_1 <_T b_2$.
(3) $a_1 <_S a_2$.

**Lemma 5.7.** *Let* $((a_1, b_1), (a_2, b_2))$ *and* $((a_2, b_2), (a_3, b_3))$ *be reversing pairs. Then* $((a_1, b_1), (a_3, b_3))$ *is also a reversing pair.*

**Proof.** Consider the following four conditions.

Condition 1: there is a point $d_1$ on the path $T(b_1)$ with $d_1 < b_2$ and $a_3 < d_1$ in $P$.

Condition 2: there is a point $e_3$ on the path $S(a_3)$ with $a_2 < e_3$ and $e_3 < b_1$ in $P$.

Condition 3: there is a point $d_3$ on the path $T(b_3)$ with $d_3 < b_2$ and $a_1 < d_3$ in $P$.

Condition 4: there is a point $e_1$ on the path $S(a_1)$ with $a_2 < e_1$ and $e_1 < b_3$ in $P$.

Considering paths that witness $a_2 < b_1$ and $a_3 < b_2$ in $P$, we see that one or both of Conditions 1 and 2 must hold. Both of these conditions imply that $a_3 < b_1$ in $P$.

Similarly, one or both of Conditions 3 and 4 must hold. Both of these conditions imply that $a_1 < b_3$ in $P$. So $((a_1, b_1), (a_3, b_3))$ is a reversing pair. □

We illustrate in Fig. 6 the situation where only Conditions 2 and 3 of Lemma 5.7 are valid.

A sequence $((a_1, b_1), (a_2, b_2), \ldots, (a_t, b_t))$ of critical pairs from $\text{Crit}^*(R)$ is called a *reversing sequence* if each consecutive pair in the sequence is a reversing pair. In view of the Lemma 5.7, this implies that $((a_{i_1}, b_{i_1}), (a_{i_2}, b_{i_2}))$ is a reversing pair for all $1 \le i_1 < i_2 \le t$.

The next result is similar in flavor to Lemma 5.6.

**Lemma 5.8.** *The length of a reversing sequence* $((a_1, b_1), (a_2, b_2), \ldots, (a_t, b_t))$ *is bounded as a function of h.*

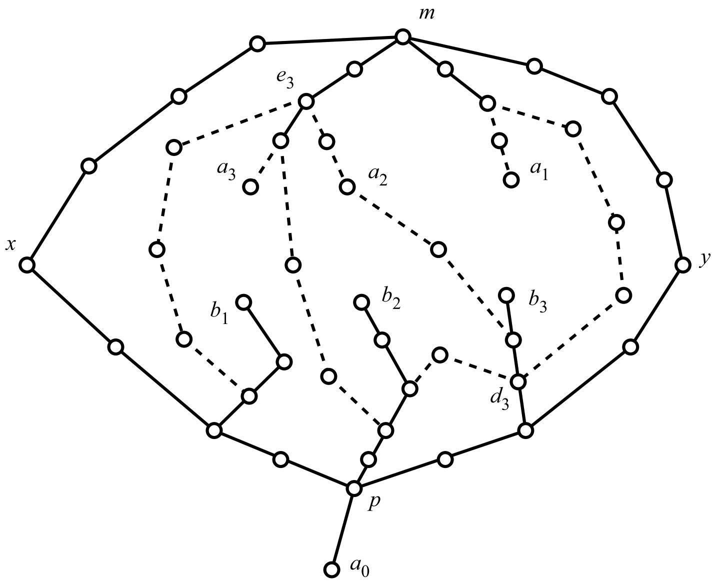


**Fig. 6.** Transitivity of reversing pairs.

**Proof.** Again, we use Ramsey theory but this time, we color triples. First, using Lemma 5.7, for each triple, we choose one of the four conditions which it satisfies; also record the associated path length. For example, if the triple satisfies Condition 3, we record $|T(d_3, b_3)|$. Accordingly, the triples are colored using at most $4h$ colors. We show that we get a contradiction if there is a monochromatic sequence of length 4. To make the argument concrete, we assume that, after relabeling, we have a monochromatic subsequence $((a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4))$ with all four triples satisfying Condition 3. Considering the triple $\{2, 3, 4\}$, there is a point $d_4$ on $T(b_4)$ with $d_4 < b_2$ and $a_2 < d_4$ in $P$. Now consider the triple $\{1, 2, 4\}$. Then there is a point $d'_4$ on $T(b_4)$ with $d'_4 < b_2$ and $a_1 < d'_4$ in $P$. Since $|T(d_4, b_4)| = |T(d'_4, b_4)|$, we conclude that $d_4 = d'_4$. But now we have $a_2 < d_4 = d'_4 < b_2$, which is false. Similar contradictions are reached in other cases. □

In view of this lemma, we add to the signature of a critical pair $(a, b)$ in Crit* the largest integer $t$ for which there is a reversing sequence of length $t$ starting with $(a, b)$. We also store the largest integer $t'$ for which there is a reversing sequence ending with $(a, b)$.

We remind the reader that there can be arbitrarily many dangerous pairs for which all the parameters identified thus far as part of the signature are constant.

It is natural to say that pairs in Crit* are *left-dangerous* if $h_j > h_{j+1}$, *center-dangerous* if $h_j = h_{j+1}$ and *right-dangerous* if $h_j < h_{j+1}$. In Fig. 4, $(a, b_4)$ is left-dangerous, $(a, b_5)$ is center-dangerous and $(a, b_6)$ is right-dangerous.

### *5.4. Center-dangerous critical pairs*

We have already done all that is required for center-dangerous critical pairs—although this may not be at all obvious at the moment.

**Lemma 5.9.** *The set of all center-dangerous critical pairs with the same signature is reversible.*

**Proof.** The proof involves a modification of the classic proof used by Erdős and Szekeres to show that any sequence of $nm + 1$ distinct real numbers has either an increasing subsequence of length $n + 1$ or a decreasing subsequence of length $m + 1$.

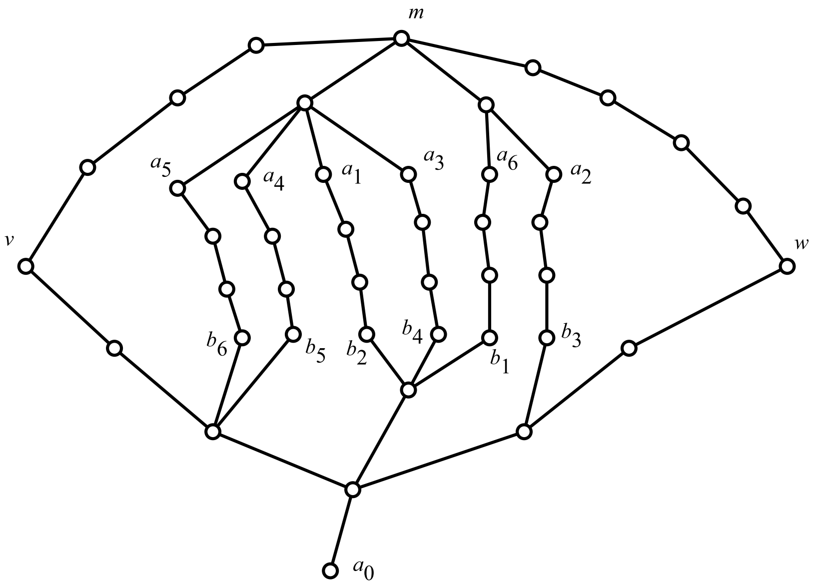


**Fig. 7.** A strict alternating cycle in the center region.

Suppose the lemma is false and that there is a strict alternating cycle $\{(a_i, b_i) : 1 \leq i \leq k\}$ of center-dangerous pairs having the same signature. For each $i$, $b_{i+1}$ is in the interior of the region $R_{i+1}$. Since $a_i \leq b_{i+1}$ in the partial order $P$, a path in the oriented graph $G$ from $a_i$ to $b_{i+1}$ cannot pass through any vertex on the boundary of $\mathcal{R}_{i+1}$, and it follows that $a_i$ is also in the interior of $\mathcal{R}_{i+1}$. However, this implies that $\mathcal{R}_i \subseteq \mathcal{R}_{i+1}$. Since this statement must hold cyclically, we must have $\mathcal{R}_i = \mathcal{R}_{i+1}$ for all $i$, and we denote this common region as $\mathcal{R}$. Of course, this also implies that there are points $m$, $x$ and $y$, so that $x = x_i$, $y = y_i$ and $m = m_i$ for all $i$.

We illustrate this situation in Fig. 7 where we show an alternating cycle of length 6. In composing this figure, we took for each $i$ an arbitrary oriented path $Q_i = Q_i(a_i, b_{i+1})$ witnessing that $a_i \leq b_{i+1}$ in $P$. In view of the properties of a strict alternating cycle, the paths $Q_1, Q_2, \ldots, Q_k$ are pairwise disjoint. Our figure may serve to suggest that the manner in which these paths intersect with the trees $T$ and $S$ is overly restricted. However, this much is certain. When $i_2 \neq i_1 + 1$, the oriented path $Q_i$ cannot intersect $T(b_{i_2})$. Also, when $i_2 \neq i_1 - 1$, the oriented path $Q_i$ cannot intersect $S(a_{i_2})$.

We consider the linear order $<_T$ restricted to $\{b_1, b_2, \ldots, b_k\}$ and let $b_i$ be the $<_T$-least element. If $k = 2$, then $((a_1, b_1), (a_2, b_2))$ is a reversing sequence. This would imply that $(a_1, b_1)$ and $(a_2, b_2)$ do not have the same signature. So we may assume that $k \geq 3$. Now consider the paths $Q_i$, $Q_{i-1}$ and $Q_{i-2}$.

At this point, we have two cases, depending on the order of $b_{i+1}$ and $b_{i-1}$ in the tree $T$. We consider first the case where $b_{i-1} <_T b_{i+1}$, as this is the situation in our figure. The argument for the other case is much the same.

Now let $t'$ be the maximum length of a reversing sequence ending in $(a_i, b_i)$. Since $((a_{i-2}, b_i), (a_{i-1}, b_{i-1}))$ is a reversing pair, we know that $t' \geq 2$. It suffices to show that, given any reversing sequence ending with $(a_i, b_i)$, we can (1) delete $(a_i, b_i)$ from the end of the sequence and (2) add the reversing pair $((a_{i-2}, b_i), (a_{i-1}, b_{i-1}))$ in its place to form a reversing sequence of length $t' + 1$ ending at $(a_{i-1}, b_{i-1})$. To this end, let $(a, b)$ be the pair immediately before $(a_i, b_i)$ in the sequence. We need only show that $((a, b), (a_{i-2}, b_i))$ is a reversing pair, but this is a straightforward argument following along the same lines as the proof of Lemma 5.7. □

We are left with handling the left-dangerous and right-dangerous pairs, a challenge which will be much more formidable. In the next section, we treat the left-dangerous pairs, noting that the arguments for right-dangerous pairs are dual.

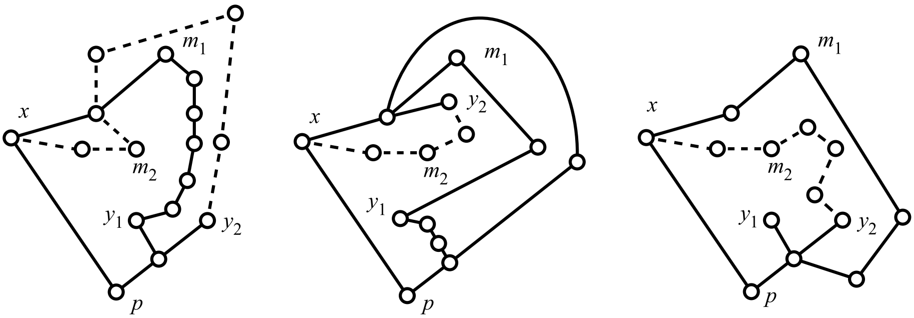


**Fig. 8.** Left-consistent pairs: three types.

## 6. Left-dangerous critical pairs

We pause to develop one additional parameter, which is a slight generalization of the left-to-right property studied previously. With the additions to signatures made using this parameter, we will then show that there are no strict alternating cycles among critical pairs with the same signature.

Let $x$ be a vertex in $G$ and consider the subfamily of all critical pairs $(a, b)$ in Crit$^*$ for which $x = x_j(a)$. In view of Proposition 5.4, the paths $L(m(a), x)$ with $a$ the first coordinate of a critical pair in this subfamily form a tree which we denote $T_x$. So we will write $m(a) < m(a')$ in $T_x$ when a clockwise scan of $T_x$, starting with $x$ as root, discovers $m(a)$ before $m(a')$.

Now let $(a_1, b_1)$ and $(a_2, b_2)$ be critical pairs from Crit$^*$ with $x = x_1 = x_2$. We say that $((a_1, b_1), (a_2, b_2))$ is a *left-consistent* pair when $m_1 < m_2$ in $T_x$ and $y_1 <_T y_2$. Note that this definition requires $m_2$ to be in the interior of $\mathcal{R}_1$. Nevertheless, as we illustrate in Fig. 8, there are essentially three different configurations for a left-consistent pair. The first arises when $y_2$ is in the exterior of $\mathcal{R}_1$. The second arises when $y_2$ is not in the exterior of $\mathcal{R}_1$ and $T(y_2)$ intersects $L(m_1, x)$. The third arises when $y_2$ is in the interior of $\mathcal{R}_1$ and $R(m_1, y_1)$ intersects $T(y_2)$.

A sequence $((a_1, b_1), (a_2, b_2), \ldots, (a_t, b_t))$ of critical pairs with $x = x_i$ for all $i = 1, 2, \ldots, t$ is called a *left-consistent sequence* when $((a_i, b_i), (a_{i+1}, b_{i+1}))$ is a left-consistent pair for each $i = 1, 2, \ldots, t$. Trivially, the notion of left-consistency is transitive.

**Lemma 6.1.** *For a fixed special point $x$, the length of a left-consistent sequence $((a_1, b_1), (a_2, b_2), \ldots, (a_t, b_t))$ is bounded as a function of $h$.*

**Proof.** We use Ramsey theory for a third time. For each ordered pair $(i_1, i_2)$ with $1 \leq i_1 < i_2 \leq t$, we consider the left consistent pair $((a_{i_1}, b_{i_1}), (a_{i_2}, b_{i_2}))$. First, we record which of the three configurations (see Fig. 8) is applicable. Second, we record all relevant path lengths. For example, if the pair is of the third type, we choose a point $z$ from $T(y_2) \cap R(m_1, y_1)$ and record $|T(z)|$ and $|R(m_1, z)|$.

We now show that there cannot be a monochromatic set of size 3. We provide the details when all pairs are of the third type. The argument in the other two cases is very similar. Suppose that after relabeling, $((a_1, b_1), (a_2, b_2), (a_3, b_3))$ is a monochromatic set of size 3 with all three pairs of the third type. Then considering the pair $(1, 2)$, there is a point $z_1$ on $T(y_2) \cap R(m_1, y_1)$. Then considering the pair $(1, 3)$, there is a point $z_3$ on $T(y_3) \cap R(m_1, y_1)$. Since the path lengths are constant, we see that $z_1 = z_3$. This forces $a_2 \leq y_1$ in $P$ which violates the unimodal requirements for $a_2$. $\square$

As before, we add to the signature of a critical pair $(a, b)$ the maximum integer $t$ for which there exists a left-consistent sequence of length $t$ starting with $(a, b)$.

We have now assembled all the tools we need to complete the argument. We assume that we have a strict alternating cycle $\{(a_i, b_i) : 1 \leq i \leq k\}$ with all pairs having the same signature and argue to a contradiction. We start by developing several properties of the critical pairs in the cycle.

**Lemma 6.2.** *If $x_{i-1} <_T x_i$, then $y_{i-1} <_T x_i$.*

**Proof.** We assume that $y_{i+1} >_T x_i$ and argue to a contradiction. We start by noting that the unimodal sequence properties for $a_i$ force it to be incomparable with $x_{i-1}$. If there is a point $u$ from $T(x_i)$ with $a_{i-1} < u$ in $P$, then there is special point $s_1 \in \text{Spec}(a_{i-1})$ with $s_1$ on the path $T(u)$. This implies that $h(s_1) \leq h_j$. But we also know that $x_{i-1} <_T s_1 < y_{i-1}$, which violates the requirements for the unimodal sequence of special points for $a_{i-1}$.

Now let $Q_1(a_{i-1}, x_{i-1})$ and $Q_2(a_{i-1}, b_i)$ be oriented paths in $G$. These two paths show that regardless of whether $a_{i-1}$ is the interior of $\mathcal{R}_i$ or in the exterior, we always have $\text{Spec}(a_{i-1}) \cap T(y_i) \neq \emptyset$.

*Case* 1. There is a special point $s_1 \in \text{Spec}(a_{i-1}) \cap T(y_i)$ with $s_1 < x_{i-1}$ in $P$.

In this case, it is clear that $s_1 \neq y_i$. Let $z$ be the last point of $Q_1(a_{i-1}, x_{i-1})$ which belongs to $T(y_i)$. Then the oriented path $Q_1(z, x_{i-1})$ and $T(x_{i-1}, z)$ form a region $\mathcal{F}$ which properly contains $\mathcal{R}_i$. It is clear that both $m_i$ and $a_i$ belong to the interior of $\mathcal{F}$.

Now let $c = \min\{h(s) : s \in \text{Spec}(a_{i-1}), s < x_{i-1} \text{ in } P\}$ and let $s_2$ be the $<_T$-largest point of $\text{Spec}(a_{i-1})$ with $h(s_2) = c$. Since $h(s_2) < h_{j+1}$, there is a point $s_3$ from $\text{Spec}(a_i)$ with $h(s_3) = h(s_2) = c$. If $s_2 = s_3$, then $a_i < x_{i-1}$ in $P$. The contradiction shows that $s_2 \neq s_3$. If $s_2 <_T s_3$, then the path $Q_1(z, x_{i-1})$ cannot intersect $T(s_3)$ for this contradict the value of $c$. It follows that $s_3$ is outside the region $\mathcal{F}$. Now an oriented path $Q_3(a_i, s_3)$ must cross the boundary of $\mathcal{F}$ which would again force $a_i < x_{i-1}$ in $P$. We conclude that $y_i <_T s_3 <_T s_2$. As before the path $Q_1(z, x_{i-1})$ cannot contain $s_3$, so it cannot intersect $T(s_3)$. This places $s_3$ in the exterior of $\mathcal{F}$, which forces $Q_3(a_i, s_3)$ to intersect the boundary of $\mathcal{F}$, which it cannot do. The contradiction completes the proof of Case 1.

*Case* 2. There is no special point $s_1 \in \text{Spec}(a_{i-1}) \cap T(y_i)$ with $s_1 < x_{i-1}$ in $P$.

In this case, $a_{i-1}$ is in the exterior of $\mathcal{R}_i$. Furthermore, any oriented path from $a_{i-1}$ to $x_{i-1}$ must not contain any points from the boundary of $\mathcal{R}_i$. Let $u$ be the last common point of $Q_1(a_{i-1}, x_{i-1})$ and $Q_2(a_{i-1}, b_i)$. Let $z$ be the first point on the oriented path $Q_2(u, b_i)$ which belongs to $T(y_i)$. Then $z \neq y_i$. Also, $T(x_{i-1}, z)$, $Q_2(u, z)$ and $Q_1(u, x_{i-1})$ form the boundary of a region $\mathcal{F}$ properly containing $\mathcal{R}_i$. As before both $m_i$ and $a_i$ are in the interior of $\mathcal{F}$.

We already know that there is a point from $\text{Spec}(a_{i-1})$ on $T(y_i)$, but now we also know that this special point is less than $u$ in $P$. Now let $c = \min\{h(s) : s \in \text{Spec}(a_{i-1}), s < u \text{ in } P\}$ and let $s_2$ be the $<_T$-largest point of $\text{Spec}(a_{i-1})$ with $h(s_2) = c$. Since $h(s_2) < h_{j+1}$, there is a point $s_3$ from $\text{Spec}(a_i)$ with $h(s_3) = h(s_2) = c$. Clearly, $s_3$ is not on the boundary of $\mathcal{F}$. If $s_3$ is in the exterior of $\mathcal{F}$, then any oriented path from $a_i$ to $s_3$ would have to intersect one of $Q_1(u, x_{i-1})$ and $Q_2(u, z)$. The first statement implies that $a_i < x_{i-1}$ in $P$. The second implies that $a_i < z$ in $P$. Both implications are false, so we may conclude that $s_3$ is in the interior of $\mathcal{F}$.

The path $Q_2(u, z)$ cannot intersect $T(s_3)$. Similarly, the path $Q_1(u, x_{i-1})$ cannot intersect $T(s_3)$. This forces $s_3$ to be in the exterior of $\mathcal{F}$. Now any oriented path from $a_i$ to $s_3$ must intersect the boundary of $\mathcal{F}$, which it cannot do. With this observation, the proof of Case 2 is complete. □

We illustrate one of the cases in the preceding lemma in Fig. 9. In this picture, we show $a_{i-1}$ in the interior of $\mathcal{R}_i$ and $y_i <_T s_2$.

**Lemma 6.3.** *If $x_i <_T x_{i-1}$, then $y_{i-1} \leq_T y_i$.*

**Proof.** We argue by contradiction. Suppose that $y_i <_T y_{i-1}$. We claim if $a_{i-1}$ is in the exterior of $\mathcal{R}_i$, then an oriented path from $a_{i-1}$ to $b_i$ would have to intersect the boundary of $\mathcal{R}_i$. This would force either $a_{i-1} \leq x_i$ in $P$ or $a_{i-1} \leq y_i$ in $P$. Both statements violate the unimodal requirements for $a_{i-1}$. We conclude that $a_{i-1}$ is in the interior of $\mathcal{R}_i$. Similarly, if $u \geq a_{i-1}$ in $P$, then $u$ must also belong to the interior of $\mathcal{R}_i$. In particular, all points of $\text{Spec}(a_{i-1})$ are in the interior of $\mathcal{R}_i$. Furthermore, if $s \in \text{Spec}(a_{i-1})$ and $h(s) \leq h(y_{i-1})$, then $y_{i-1} \leq_T s$ and the path $T(s)$ intersects the boundary of $\mathcal{R}_i$ at a point $x$ with $a_i \leq x$ in $P$. It follows that there is a special point $s'$ from $\text{Spec}(a_i)$ with $s'$ on the path $T(s)$, $s'$ not in the interior of $\mathcal{R}_i$ and $h(s') \leq h(x) < h(s)$.

We illustrate this last assertion in Fig. 10. Furthermore, applying the assertion to $y_{i-1}$, we see that there is a special point $s_0$ on $T(y_{i-1})$ with $s_0$ not in the interior of $\mathcal{R}_i$ and $h(s_0) < h(y_{i-1})$.

Now let $C$ consist of all $s \in \text{Spec}(a_i)$ satisfying (a) $s$ is not in the interior of $\mathcal{R}_i$, (b) $h(s) \leq h(y_{i-1})$ and (c) there is some point $z$ on the boundary of $\mathcal{R}_i$ for which $s \leq z$ in $P$. The set $C$ is non-empty since $s_0 \in C$. Now let $c = \min\{h(s) : s \in C\}$ and let $s_1$ be any element of $C$ with $h(s_1) = c$.

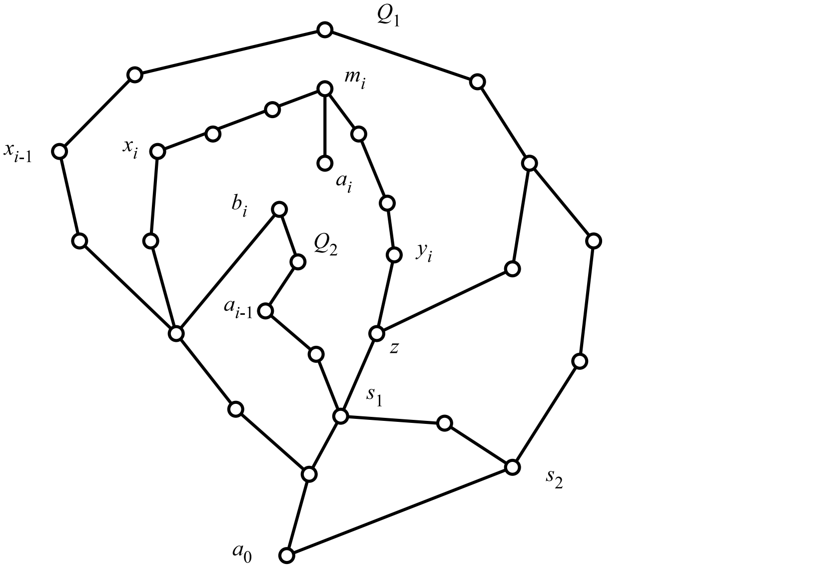


**Fig. 9.** Left-going special points.

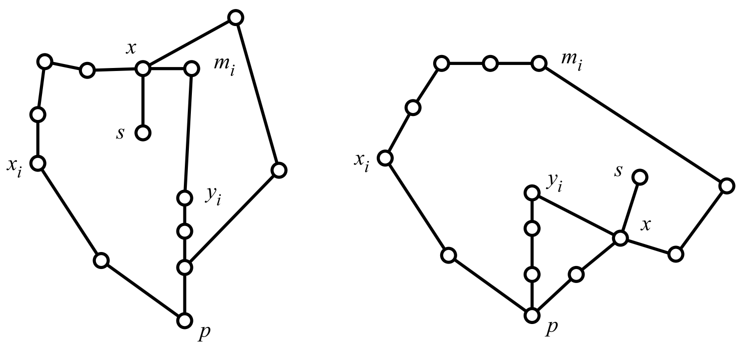


**Fig. 10.** Right-going special points.

Now let $s_2$ be a point from $\text{Spec}(a_{i-1})$ with $h(s_2) = h(s_1)$. Then $s_2$ is in the interior of $\mathcal{R}_i$ so, using analysis similar to that above, there is a point $s_3 \in \text{Spec}(a_i)$ so that $s_3$ is not in the interior of $\mathcal{R}_i$ and $s_3$ is on $T(s_2)$. This is a contradiction since $h(s_3) < h(s_2) = h(s_1)$. $\square$

Consider the set of (not necessarily) distinct points $\{y_i : 1 \leq i \leq k\}$ and let $I_1 = \{i : y_i \leq_T y_{i'}$ for all $i'$ with $1 \leq i' \leq k\}$. Then set $I_2 = \{i \in I_1 : x_i \geq x_{i'}$ for all $i' \in I_1\}$. Then let $i$ be any integer from $I_2$.

We claim that $x_{i-1} = x_i$, for if $x_{i-1} <_T x_i$, then $y_{i-1} <_T x_i <_T y_i$ by Lemma 6.2, which violates the fact that $i \in I_1$. On the other hand, if $x_{i-1} >_T x_i$, then we know by Lemma 6.3 that $y_{i-1} \leq_T y_i$, which shows that $i \notin I_2$.

Now we know that $x_{i-1} = x_i$, we consider whether $y_{i-1}$ is distinct from $y_i$. Suppose first that $y_{i-1} \neq y_i$ which requires $y_i <_T y_{i-1}$. If $m_{i-1} < m_i$ in $T_x$, then we consider an oriented path $Q(a_{i-1}, b_i)$. If this path intersects the boundary of $\mathcal{R}_i$ at any point $u$ with $m_i \leq u$ in $P$, then $a_i < b_i$ in $P$, which is false. But if $Q(a_{i-1}, b_i)$ intersects the boundary of $\mathcal{R}_i$ at a point $u$ with $u \parallel m_i$, then the unimodal sequence requirements for $a_{i-1}$ are violated. We conclude that $a_{i-1}$ is in the interior of $\mathcal{R}_i$. Now consider an oriented path $Q(a_{i-1}, m_i)$. Clearly, this path must intersect the boundary of $\mathcal{R}_i$ and we obtain a contradiction regardless of how this occurs.

It follows that $m_i < m_{i-1}$ in $T_x$. But this implies that $((a_{i-1}, b_{i-1}), (a_i, b_i))$ is a left-consistent pair. This implies that these two pairs do not have the same signature.

We are left to conclude that $y_{i-1} = y_i$. Thus $i - 1$ also belongs to $I_2$. Proceeding around the cycle, we see that we must have $I_2 = \{1, 2, \ldots, k\}$. This implies that there are points $x$ and $y$ so that $x = x_i$ and $y = y_i$ for all $i = 1, 2, \ldots, k$. It follows that for each $i = 1, 2, \ldots, k$, one of $\mathcal{R}_i$ and $\mathcal{R}_{i-1}$ is a subset of the other. Now suppose that for some $i$, the region $\mathcal{R}_i$ is properly contained in $\mathcal{R}_{i-1}$. Then $m_i \neq m_{i-1}$. If $m_{i-1} < m_i$ in $T_x$, then $m_{i-1}$ is in the exterior of $\mathcal{R}_i$. On the other hand, we must have $a_{i-1}$ in the interior of $\mathcal{R}_i$. However, this implies that an oriented path from $a_{i-1}$ to $m_{i-1}$ must intersect the boundary of $\mathcal{R}_i$ which cannot happen.

We conclude that $m_i < m_{i-1}$ in $T_x$, which is impossible when $\mathcal{R}_i$ is properly contained in $\mathcal{R}_{i-1}$. We are left to conclude that $\mathcal{R}_{i-1}$ is always contained in $\mathcal{R}_i$. Since we have an alternating cycle, we find that there is a fixed region $\mathcal{R}$ so that $\mathcal{R} = \mathcal{R}_i$ for all $i = 1, 2, \ldots, k$. Now the contradiction comes for the subsection on fixed regions. And with this observation, we have finished the proof of our main theorem in the special case that there is some minimal element $a_0$ which is less than all maximal elements.

## 7. The general case

As promised, we can quickly dispense with the general case. Fix an arbitrary minimal element $a_0$. Set $B_1 = \{b \in \max(P) : a_0 < b \text{ in } P\}$ and $A_1 = \{a \in \min(P) - \{a_0\}$: there is some $b \in B_1$ with $a < b$ in $P\}$. Then define recursively subsets of $A$ and $B$ as follows:

(1) $B_{i+1}$ consists of all maximal elements of $B - (B_1 \cup B_2 \cup \cdots \cup B_i)$ for which there is some $a \in A_i$ with $a < b$ in $P$.
(2) $A_{i+1}$ consists of all minimal elements of $A - (\{a_0\} \cup A_1 \cup A_2 \cup \cdots \cup A_i)$ for which there is some $b \in B_{i+1}$ with $a < b$ in $P$.

**Proposition 7.1.** *The set* $\{(a, b) \in \text{Crit}^*(P) : a \in A_{i_1}, b \in B_{i_2}, i_1 \geq 1 + i_2\}$ *is reversible. Also, the set* $\{(a, b) \in \text{Crit}^*(P) : a \in A_{i_1}, b \in B_{i_2}, i_2 \geq 2 + i_1\}$ *is reversible.*

**Proposition 7.2.** *For each* $i \geq 1$, *the set* $\text{Crit}^*(P, \textit{Left})$ *consisting of all critical pairs* $(a, b)$ *with* $a \in A_i$ *and* $b \in B_i$ *can be partitioned into a bounded number of reversible sets. Also, for each* $i \geq 1$, *the set* $\text{Crit}^*(P, \textit{Right})$ *consisting of all critical pairs* $(a, b)$ *with* $a \in A_i$ *and* $b \in B_{i+1}$ *can be partitioned into a bounded number of reversible sets.*

**Proof.** The proof for $\text{Crit}^*(P, \text{Left})$ and $i = 1$ is just what we have done prior to this section. All other cases can be reduced to this base case by taking graph minors. To be more precise, we can contract all points in $(A_1 \cup B_1) \cup (A_2 \cup B_2) \cup \cdots \cup (A_{i-1} \cup B_{i-1}) \cup \{a_0\}$ to a single point which is less than all points of $B_i$. The other case is dual. □

We are now ready to claim the entire main theorem. This follows from the fact that any strict alternating cycle in $\text{Crit}^*(P, \text{Left})$ is associated with a single value of $i$. A similar remark applies for $\text{Crit}^*(P, \text{Right})$.

## 8. Concluding remarks and acknowledgment

Returning to Fig. 2, we previously discussed relabeling the element marked with a 1 as a 0. Relabel it again as a minimal element $a_0$. Then add a new point in the center of the figure. Connect it to $a_1, a_2, \ldots, a_8$ and label it as $b_0$. The resulting diagram is a drawing without edge crossings of the cover graph of height seven containing the standard example $S_9$ as a subposet. In general, this shows that $c_h \geq h + 2$ and this might even be the right answer.

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