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An extremal problem on crossing vectors



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ABSTRACT

For positive integers w and k , two vectors A and B from \mathbb{Z}^w are called k -crossing if there are two coordinates i and j such that $A[i] - B[i] \geq k$ and $B[j] - A[j] \geq k$. What is the maximum size of a family of pairwise 1-crossing and pairwise non- k -crossing vectors in \mathbb{Z}^w ? We state a conjecture that the answer is k^{w-1} . We prove the conjecture for $w \leq 3$ and provide weaker upper bounds for $w \geq 4$. Also, for all k and w , we construct several quite different examples of families of desired size k^{w-1} . This research is motivated by a natural question concerning the width of the lattice of maximum antichains of a partially ordered set.

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1. Introduction

We deal with vectors in \mathbb{Z}^w , which we call just *vectors*. The i th coordinate of a vector $A \in \mathbb{Z}^w$ is denoted by $A[i]$, for $1 \leq i \leq w$. The product ordering on \mathbb{Z}^w is defined by setting $A \leq B$ for $A, B \in \mathbb{Z}^w$ whenever $A[i] \leq B[i]$ for every coordinate i . When $k \geq 1$, we say that vectors A and B from \mathbb{Z}^w are k -crossing if there are coordinates i and j for which $A[i] - B[i] \geq k$ and $B[j] - A[j] \geq k$. Thus \mathcal{A} is an antichain in \mathbb{Z}^w if and only if any two distinct vectors from \mathcal{A} are 1-crossing. A family of vectors in \mathbb{Z}^w is k -crossing-free if it contains no two k -crossing vectors.

For positive integers k and w , let $f(k, w)$ denote the maximum size of a subset of \mathbb{Z}^w with any two vectors being 1-crossing but not k -crossing. In other words, $f(k, w)$ is the maximum size of a k -crossing-free antichain in \mathbb{Z}^w . Note that an antichain of vectors in \mathbb{Z}^w with $w \geq 2$ without the restriction that no two vectors are k -crossing can have infinite size (e.g. $\{(k, -k): k \in \mathbb{Z}\}$ for $w = 2$). Similarly, there are infinite k -crossing-free families of vectors in \mathbb{Z}^w which are not antichains (e.g. $\{(k, k): k \in \mathbb{Z}\}$ for $w = 2$).

Determining the value of $f(k, w)$ is the main focus of this paper. The following striking simple conjecture was formulated in 2010 and never published, so we state it here with the kind permission of its authors.

Conjecture 1 (Felsner, Krawczyk, Micek). *For all $k, w \geq 1$, we have*

$$f(k, w) = k^{w-1}.$$

At first, it is even not clear whether $f(k, w)$ is bounded for all k and w . We prove the conjecture for $1 \leq w \leq 3$ and provide lower (matching the conjectured value) and upper bounds on $f(k, w)$ for $w \geq 4$. Still, we are unable to resolve the conjecture in full generality.

Theorem 2. *For $1 \leq w \leq 3$ and $k \geq 1$, we have*

$$f(k, w) = k^{w-1}.$$

Theorem 3. *For $w \geq 4$ and $k \geq 1$, we have*

$$k^{w-1} \leq f(k, w) \leq \min\{k^w - k^2(k-1)^{w-2}, \lceil \frac{w}{3} \rceil k^{w-1}\}.$$

The remainder of this paper is organized as follows. We start, in the next section, by a brief discussion of problems in partially ordered sets that initiated this research. Section 3 is devoted to the proof of [Theorem 3](#) and the lower bound of [Theorem 2](#). The upper bound of [Theorem 2](#) is proved in Section 4. In Section 5, we propose another conjecture, which is at first glance more general but in fact equivalent to [Conjecture 1](#). Concluding in Section 6, we provide examples of families witnessing $f(k, w) \geq k^{w-1}$ with

a discussion why the full resolution of the conjecture seems to be difficult. We also present a proof of the conjecture for families of vectors with a single coordinate differentiating all vectors in the family and another argument for ranked families of vectors, that is, families in which the coordinates of every vector sum up to the same value.

2. Background motivation

Let $\mathcal{M}(P)$ denote the family of all maximum antichains (that is, antichains of maximum size) in a finite poset P . The family $\mathcal{M}(P)$ is partially ordered by setting $A \leq B$ when for every $a \in A$ there is $b \in B$ with $a \leq b$ in P , or equivalently, when for every $b \in B$ there is $a \in A$ with $a \leq b$ in P . The family $\mathcal{M}(P)$ equipped with this partial order forms a distributive lattice [3], and every finite distributive lattice is isomorphic to $\mathcal{M}(P)$ for some poset P [8]. In the following, we are concerned with the order structure of $\mathcal{M}(P)$, in particular its width.

For positive integers k_1, \dots, k_n , let $\mathbf{k}_1 + \dots + \mathbf{k}_n$ denote the poset consisting of n pairwise disjoint chains of sizes k_1, \dots, k_n with no comparabilities between points in distinct chains. For a positive integer k , let $\mathcal{P}(k)$ denote the class of posets containing no subposet isomorphic to $\mathbf{k} + \mathbf{k}$. The posets in $\mathcal{P}(1)$ are just the chains, while $\mathcal{P}(2)$ is exactly the class of interval orders [6,9]. For positive integers k and w , let $\mathcal{P}(k, w)$ denote the subclass of $\mathcal{P}(k)$ consisting of posets of width at most w .

Recently, several results in combinatorics of posets showed that problems that are difficult or even impossible to deal with for all posets of bounded width become much easier when only posets from $\mathcal{P}(k, w)$ are considered. This includes the on-line chain partitioning problem [1,4,7] and the on-line dimension problem [5].

It is easy to see that the width of $\mathcal{M}(\mathbf{k} + \mathbf{k})$ is k , and it follows from Sperner’s theorem [10] that the width of $\mathcal{M}(\underbrace{\mathbf{2} + \dots + \mathbf{2}}_w)$ is $\binom{w}{\lfloor w/2 \rfloor}$. However, it turns out that the width of $\mathcal{M}(P)$ can be bounded by a constant when the width of P is bounded and the size of a $\mathbf{k} + \mathbf{k}$ type structure in P is bounded as well.

Proposition 4. *For $k, w \geq 1$ and $P \in \mathcal{P}(k + 1, w)$, the width of $\mathcal{M}(P)$ is at most $f(k, w)$.*

Proof. By Dilworth’s theorem [2], P can be covered with w chains C_1, \dots, C_w . Each of them intersects each antichain $A \in \mathcal{M}(P)$. Enumerate the elements of each chain C_i as $c_{i,1}, \dots, c_{i,|C_i|}$ according to their order in the chain. For an antichain $A \in \mathcal{M}(P)$, define a vector $A' \in \mathbb{Z}^w$ so that $A'[i]$ is the height in C_i of the element common to both A and C_i , that is, $A \cap C_i = \{c_{i,A'[i]}\}$ for $1 \leq i \leq w$. Clearly, for every antichain $\mathcal{A} \subset \mathcal{M}(P)$, the family of vectors $\mathcal{A}' = \{A' : A \in \mathcal{A}\}$ is an antichain in \mathbb{Z}^w . Moreover, no two vectors in \mathcal{A}' are k -crossing: if $A'[i] - B'[i] = k_1 \geq k$ and $B'[j] - A'[j] = k_2 \geq k$ for some $A, B \in \mathcal{A}$ and $1 \leq i, j \leq w$, then the elements $c_{i,B'[i]}, \dots, c_{i,A'[i]}$ and $c_{j,A'[j]}, \dots, c_{j,B'[j]}$ induce a subposet of P isomorphic to $(\mathbf{k}_1 + \mathbf{1}) + (\mathbf{k}_2 + \mathbf{1})$, which contradicts the assumption that $P \in \mathcal{P}(k + 1, w)$. Therefore, we have $|\mathcal{A}| = |\mathcal{A}'| \leq f(k, w)$. \square

Conjecture 5 (*Felsner, Krawczyk, Micek*). *Let k and w be positive integers with $k \geq 2$. The maximum width of $\mathcal{M}(P)$ for a poset $P \in \mathcal{P}(k, w)$ is $(k - 1)^{w-1}$.*

This conjecture was made prior to the formulation of [Conjecture 1](#). It follows from [Proposition 4](#) that [Conjecture 1](#) implies [Conjecture 5](#). In particular, in view of [Theorem 2](#), [Conjecture 5](#) is true for $w \leq 3$. In the special case $k = 2$, since $\mathcal{P}(2, w)$ is the class of interval orders of width w , [Conjecture 5](#) states the well-known fact that the maximum antichains in an interval order form a chain. Moreover, we are able to prove that [Conjecture 5](#) for $k = 3$ and [Conjecture 1](#) for $k = 2$ are equivalent.

3. General bounds

The purpose of this section is to give the proof of [Theorem 3](#) and the lower bound of [Theorem 2](#), namely, that we have

$$\begin{aligned}
 f(k, w) &\geq k^{w-1} && \text{for } k, w \geq 1, \\
 f(k, w) &\leq \min\{k^w - k^2(k - 1)^{w-2}, \lceil \frac{w}{3} \rceil k^{w-1}\} && \text{for } w \geq 4 \text{ and } k \geq 1.
 \end{aligned}$$

Note that $f(k, 1) = k^0 = 1$ for every $k \geq 1$, as all antichains in \mathbb{Z}^1 are of size 1. Also, $f(1, w) = 1^{w-1} = 1$ for every $w \geq 1$, as in this case every pair of distinct vectors is required to be simultaneously 1-crossing and non-1-crossing.

For the lower bound, observe that the following family is a k -crossing-free antichain in \mathbb{Z}^w and has size k^{w-1} :

$$\{A \in \mathbb{Z}^w : 0 \leq A[i] \leq k - 1 \text{ for } 1 \leq i \leq w - 1, \text{ and } A[1] + \dots + A[w] = 0\}.$$

For the upper bound, we start by an easy argument that yields the bound of k^w . Let \mathcal{A} be a k -crossing-free antichain in \mathbb{Z}^w . For each vector $A \in \mathcal{A}$, let $\sigma(A)$ be the vector from $\{0, \dots, k - 1\}^w$ such that $A[i] \equiv \sigma(A)[i] \pmod{k}$ for $1 \leq i \leq w$. If $\sigma(A) = \sigma(B)$ for distinct vectors $A, B \in \mathcal{A}$, then any two coordinates i and j such that $A[i] > B[i]$ and $B[j] > A[j]$ (which must exist, as \mathcal{A} is an antichain) witness that A and B are k -crossing. It follows that σ is an injection. Since the size of the range of σ is k^w , we have $|\mathcal{A}| \leq k^w$.

We obtain better upper bounds using the following recursive formula.

Claim 6. *For $w \geq 2$ and $k \geq 1$, we have*

$$\begin{aligned}
 f(k, w) &\leq k^{w-v} f(k, v) + k^v f(k, w - v), \quad \text{for } 1 \leq v < w, \\
 f(k, w) &\leq k^{w-1} + (k - 1)f(k, w - 1).
 \end{aligned}$$

Proof. Let \mathcal{A} be a k -crossing-free antichain in \mathbb{Z}^w . When $A \in \mathcal{A}$ and $1 \leq i \leq j \leq w$, we will use $A[i, \dots, j]$ as a convenient notation for the vector $(A[i], \dots, A[j])$ in \mathbb{Z}^{j-i+1} .

Fix a residue class $(r_1, \dots, r_{w-v}) \in \{0, \dots, k-1\}^{w-v}$, and consider the family \mathcal{A}' of all vectors $A \in \mathcal{A}$ such that $A[i] \equiv r_i \pmod{k}$ for $1 \leq i \leq w-v$. For any distinct vectors $A, B \in \mathcal{A}'$, we have $A[w-v+1, \dots, w] \neq B[w-v+1, \dots, w]$, as otherwise A and B would be k -crossing. Let $\mathcal{A}'' = \{A[w-v+1, \dots, w] : A \in \mathcal{A}'\}$. The maximal vectors in \mathcal{A}'' form a k -crossing-free antichain, so there are at most $f(k, v)$ of them. Color the vectors $A \in \mathcal{A}'$ such that $A[w-v+1, \dots, w]$ is maximal in \mathcal{A}'' red and the remaining vectors in \mathcal{A}' blue. Hence there are at most $f(k, v)$ red vectors \mathcal{A}' for the fixed residue class and at most $k^{w-v} f(k, v)$ red vectors in \mathcal{A} altogether.

Now, fix a residue class $(r_{w-v+1}, \dots, r_w) \in \{0, \dots, k-1\}^v$, and consider the family \mathcal{A}' of all blue vectors $A \in \mathcal{A}$ such that $A[i] \equiv r_i \pmod{k}$ for $w-v+1 \leq i \leq w$. For any distinct vectors $A, B \in \mathcal{A}'$, we have $A[1, \dots, w-v] \neq B[1, \dots, w-v]$, as otherwise A and B would be k -crossing. Let $\mathcal{A}'' = \{A[1, \dots, w-v] : A \in \mathcal{A}'\}$. We show that \mathcal{A}'' is an antichain. Suppose to the contrary that there are two vectors $A, B \in \mathcal{A}'$ such that $A[1, \dots, w-v] < B[1, \dots, w-v]$. The vectors $A[w-v+1, \dots, w]$ and $B[w-v+1, \dots, w]$ are distinct, as otherwise we would have $A < B$, and comparable in \mathbb{Z}^v , as otherwise A and B would be k -crossing. Hence we have $A[w-v+1, \dots, w] > B[w-v+1, \dots, w]$. In particular, there is a coordinate $j \in \{w-v+1, \dots, w\}$ such that $A[j] > B[j]$, which implies $A[j] - B[j] \geq k$. By the definition of the coloring, there is a red vector $A' \in \mathcal{A}$ such that

- (i) $A'[i] \equiv A[i] \pmod{k}$ for $1 \leq i \leq w-v$, and
- (ii) $A'[w-v+1, \dots, w] > A[w-v+1, \dots, w]$.

There is a coordinate $i \in \{1, \dots, w-v\}$ such that $A'[i] < A[i]$, as otherwise we would have $A' > A$. This implies $A[i] - A'[i] \geq k$. This is a contradiction: we have $B[i] - A'[i] \geq A[i] - A'[i] \geq k$ and $A'[j] - B[j] \geq A[j] - B[j] \geq k$, so A' and B are k -crossing. We have thus shown that \mathcal{A}'' is indeed an antichain. Since \mathcal{A}'' is k -crossing-free, we have $|\mathcal{A}''| \leq f(k, w-v)$. Hence $|\mathcal{A}'| \leq f(k, w-v)$ for the fixed residue class, and there are at most $k^v f(k, w-v)$ blue vectors in \mathcal{A} altogether.

We conclude that the total number of red and blue vectors in \mathcal{A} is at most $k^{w-v} f(k, v) + k^v f(k, w-v)$, as is required for the first inequality. For the second one, if $v = 1$, then it is enough to consider residue classes of $A[w]$ modulo $k-1$ instead of k in the second part of the argument. This is because $A'[w] - B[w] \geq k$ will follow from $A[w] - B[w] \geq k-1$ and the strict inequality $A'[w] > A[w]$ (see (ii) above). \square

From the second inequality of Claim 6 and the fact that $f(k, 1) = 1$, it follows that $f(k, w) \leq k^w - (k-1)^w$. This bound is better than both k^w and wk^{w-1} . With the equality $f(k, 3) = k^2$ of Theorem 2, we get an even better bound

$$f(k, w) \leq k^w - k^2(k-1)^{w-2}, \quad \text{for } w \geq 2.$$

The first inequality of Claim 6 applied recursively with $v = 3$ and $f(k, 3) = k^2$ give an upper bound

$$f(k, w) \leq \left\lceil \frac{w}{3} \right\rceil k^{w-1}, \quad \text{for } w \geq 3.$$

4. The case $w \leq 3$

In this section, we prove [Theorem 2](#), namely, that for $1 \leq w \leq 3$ and $k \geq 1$ we have

$$f(k, w) = k^{w-1}.$$

As explained at the beginning of the previous section, the equality holds for $w = 1$ or $k = 1$. Therefore, for the rest of this section, we assume that $2 \leq w \leq 3$ and $k \geq 2$. We only need to show that $f(k, w) \leq k^{w-1}$, as the converse inequality is proved in the previous section. We start by the following easy proposition, stated for emphasis.

Proposition 7. *Let $w \geq 2$, and let \mathcal{A} be an antichain in \mathbb{Z}^w . If $S \subset \{1, \dots, w\}$, $|S| = w - 2$, and $A[i] = B[i]$ for any $A, B \in \mathcal{A}$ and every $i \in S$, then the two remaining coordinates $j, j' \in \{1, \dots, w\} \setminus S$ determine two linear orders on \mathcal{A} , one dual to the other. That is, if we set $n = |\mathcal{A}|$, then there is a labeling A_1, \dots, A_n of the vectors in \mathcal{A} such that*

$$A_1[j] < \dots < A_n[j] \quad \text{and} \quad A_1[j'] > \dots > A_n[j'].$$

In particular, A_1 and A_n are $(n - 1)$ -crossing.

It follows immediately from [Proposition 7](#) that $f(k, 2) \leq k$. Therefore, for the remainder of the argument, we fix $w = 3$ and show that $f(k, 3) \leq k^2$ for $k \geq 2$.

We say that a k -crossing-free antichain \mathcal{A} in \mathbb{Z}^w is *compressed* on the i th coordinate when $A[i] \geq 0$ for all $A \in \mathcal{A}$ and the quantity $\sum_{A \in \mathcal{A}} A[i]$ is minimized over all k -crossing-free antichains of the same size. Let \mathcal{A} be a k -crossing-free antichain in \mathbb{Z}^3 compressed on the third coordinate. It follows that $Q_3 = \{A[3] : A \in \mathcal{A}\}$ is an interval of non-negative integers starting from 0. By [Proposition 7](#), the subfamily of \mathcal{A} consisting of all vectors A with $A[3] = s$ has size at most k for any $s \geq 0$. We conclude that $|\mathcal{A}| \leq k^2$ if $|Q_3| \leq k$. Thus, for the remainder of the argument, we assume $|Q_3| > k$.

Now, we use coordinate 3 to define a directed graph D whose vertices are the vectors in \mathcal{A} . The edges in D are of two types: *short* and *long*.

- (i) D has a short edge from A to B when $A[3] - B[3] = 1$ and $A[i] \leq B[i]$ for $i \in \{1, 2\}$.
- (ii) D has a long edge from A to B when $B[3] - A[3] = k - 1$ and there is a coordinate $i \in \{1, 2\}$ for which $A[i] - B[i] \geq k$.

Claim 8. *For every $A \in \mathcal{A}$, there is a path (A_0, \dots, A_p) in D with $A_0 = A$ and $A_p[3] = 0$.*

Proof. The statement is trivial for $A \in \mathcal{A}$ with $A[3] = 0$. Suppose the conclusion of the claim is false for some vector $A \in \mathcal{A}$ with $A[3] > 0$. Let \mathcal{B} denote the subfamily of \mathcal{A} consisting of A and the vectors B in \mathcal{A} for which there is a directed path from A to B in D . Decrease coordinate 3 of each vector in \mathcal{B} by 1, thus obtaining a family \mathcal{B}' . The family $\mathcal{A}' = (\mathcal{A} \setminus \mathcal{B}) \cup \mathcal{B}'$ has the same size as \mathcal{A} , is an antichain, contains no two k -crossing

vectors, uses only non-negative coordinates, and satisfies $\sum_{A \in \mathcal{A}'} A[i] < \sum_{A \in \mathcal{A}} A[i]$. This contradicts the choice of \mathcal{A} and completes the proof of the claim. \square

Claim 9. For every $A \in \mathcal{A}$ with $A[3] \geq k$, there is a path (U_0, \dots, U_k) in D such that $U_0[3] = A[3]$ and (U_m, U_{m+1}) is a short edge in D for $0 \leq m \leq k - 1$.

Proof. Fix $A \in \mathcal{A}$ with $A[3] \geq k$. For each $U \in \mathcal{A}$ with $U[3] = A[3]$, consider the length of a shortest path $P = (U_0, \dots, U_p)$ in D from U to a vertex U_p with $U_p[3] = 0$. Of all such U and U_p , take those for which the length p of the path P is minimized. We show that the first $k + 1$ vectors on the chosen path satisfy the requirements of the claim. Suppose to the contrary that there is m with $0 \leq m \leq k - 1$ for which the edge (U_m, U_{m+1}) is long. Then $U_{m+1}[3] \geq A[3]$, and it follows that there is an integer n with $m + 1 \leq n < p$ for which $U_n[3] = A[3]$. This contradicts the choice of P and completes the proof of the claim. \square

In view of Claim 9, it is natural to refer to a path $\mathcal{P} = (U_0, \dots, U_p)$ in D as a *short path* when all edges on \mathcal{P} are short. Also, we say that the short edge (U, V) from D is *expanded in coordinate i* when $V[i] > U[i]$. Clearly, if (U, V) is a short edge in D , then it is expanded in one or both of coordinates 1 and 2 (as U and V are 1-crossing).

Let $\mathcal{A}_s = \{A \in \mathcal{A} : A[3] \equiv s \pmod{k}\}$ for $0 \leq s \leq k - 1$. To complete the proof, we show that $|\mathcal{A}_s| \leq k$ for $0 \leq s \leq k - 1$. Thus, for the remainder of the argument, we fix an integer s with $0 \leq s \leq k - 1$. Let r be the largest integer for which there is a vector $A \in \mathcal{A}$ with $A[3] = s + (r - 1)k$, and let $\mathcal{A}_s = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_r$ be the natural partition of \mathcal{A}_s such that $A[3] = s + (j - 1)k$ for each $A \in \mathcal{B}_j$. We can assume that $r \geq 2$, as otherwise the conclusion that $|\mathcal{A}_s| \leq k$ follows from Proposition 7.

For $1 \leq j \leq r$, we refer to \mathcal{B}_j as *level j* of \mathcal{A}_s . Also, for $1 \leq j \leq r - 1$, we apply Claim 9 and choose a short path \mathcal{P}_j of $k + 1$ vectors starting at a vector $X_{j+1} \in \mathcal{B}_{j+1}$ and ending at a vector $Y_j \in \mathcal{B}_j$.

Claim 10. For $2 \leq j \leq r - 1$, we have $X_j \neq Y_j$.

Proof. Suppose to the contrary that for some j with $2 \leq j \leq r - 1$ we have $X_j = Y_j$. The ending point of the short path \mathcal{P}_j is the same as the starting point of the short path \mathcal{P}_{j-1} . It follows that the union of these two paths is a short path of $2k + 1$ vectors starting at the vector X_{j+1} and ending at the vector Y_{j-1} . Denote the vectors on this path by $\mathcal{P} = (U_0, \dots, U_{2k})$, where $U_0 = X_{j+1}$ and $U_{2k} = Y_{j-1}$. We have

$$U_0[i] \leq \dots \leq U_{2k}[i] \quad \text{for } i \in \{1, 2\}.$$

Furthermore, for $0 \leq m \leq 2k - 1$, the short edge (U_m, U_{m+1}) is expanded in some coordinate $i \in \{1, 2\}$. Since there are $2k$ short edges on \mathcal{P} , at least k of them are expanded in some coordinate $i \in \{1, 2\}$. It follows that $U_{2k}[i] - U_0[i] \geq k$. Since $U_0[3] - U_{2k}[3] = 2k$,

we conclude that U_0 and U_{2k} are k -crossing. This contradiction completes the proof of the claim. \square

Let j be an integer with $1 \leq j \leq r$. Since $A[3] = s + (j - 1)k$ for all $A \in \mathcal{B}_j$, we know from Proposition 7 that (a) each of the first two coordinates determines a linear order on \mathcal{B}_j , and (b) these two linear orders are dual. In particular, if $2 \leq j \leq r - 1$, then there is a unique $i \in \{1, 2\}$ for which $X_j[i] > Y_j[i]$.

Now let $i \in \{1, 2\}$. An interval $B = [p, t]$ of consecutive integers from $[1, r - 1]$ is called a *block of type i* when the following conditions are satisfied:

- (i) $p = 1$ or $X_p[i] < Y_p[i]$;
- (ii) $X_j[i] > Y_j[i]$ for all $j \in (p, t]$;
- (iii) $t = r - 1$ or $X_{t+1}[i] < Y_{t+1}[i]$.

The blocks of type i form a partition of the integer interval $[1, r - 1]$. In particular, every $j \in [1, r - 1]$ belongs to two blocks, one of each type. Moreover, for every $j \in [1, r - 2]$, there is a unique i such that j and $j + 1$ belong together to a block of type i . This implies that there are exactly r blocks altogether. When $r = 2$, the singleton set $\{1\}$ is a block of both types, as the three conditions listed above are satisfied vacuously, and it is counted twice.

Choose j with $1 \leq j \leq r - 1$. Let $\mathcal{P}_j = (U_0, \dots, U_k)$. For $i \in \{1, 2\}$, let B_i be the block of type i containing j , that is, $B_i = [p_i, t_i]$ with $p_i \leq j \leq t_i$. When a short edge (U_m, U_{m+1}) with $0 \leq m \leq k - 1$ is expanded in coordinate i , we say that (U_m, U_{m+1}) is *expanded in B_i* . Each of the short edges (U_m, U_{m+1}) , for $0 \leq m \leq k - 1$, is expanded in at least one of B_1 and B_2 .

Now, choose j with $1 \leq j \leq r$. Let U and V be distinct vectors in \mathcal{B}_j that occur consecutively in the two linear orders induced by coordinates 1 and 2. We say that the pair (U, V) *contributes a space* to a block $B = [p, t]$ of type i when one of the following three conditions is satisfied:

- (i) $j = p$ and $U[i] > V[i] \geq Y_j[i]$;
- (ii) $p + 1 \leq j \leq t$ and $X_j[i] \geq U[i] > V[i] \geq Y_j[i]$;
- (iii) $j = t + 1$ and $X_j[i] \geq U[i] > V[i]$.

Claim 11. *Let j be an integer with $1 \leq j \leq r$. If $U, V \in \mathcal{B}_j$ are consecutive in the linear orders on \mathcal{B}_j determined by coordinates 1 and 2, then exactly one of (U, V) and (V, U) contributes a space to a block and that block is unique.*

Proof. Assume without loss of generality that $U[1] > V[1]$ and $V[2] > U[2]$.

Suppose first that $j = 1$. If $U[1] > V[1] \geq Y_1[1]$, then (U, V) contributes a space to the block of type 1 containing 1. Otherwise, we have $V[2] > U[2] \geq Y_1[2]$ and (V, U) contributes a space to the block of type 2 containing 1.

Also, each of the $(r - 1)k$ short edges on the paths $\mathcal{P}_1, \dots, \mathcal{P}_{r-1}$ is expanded in at least one block. Thus, by [Claim 12](#), we have

$$\sum_{j=1}^r (b_j - 1) + (r - 1)k \leq r(k - 1).$$

On the other hand, we have

$$\sum_{j=1}^r (b_j - 1) = |\mathcal{A}_s| - r.$$

Composing the two, we obtain $|\mathcal{A}_s| \leq k$, which completes the proof of [Theorem 2](#).

5. Generalization

For $w \geq 1$ and $1 \leq k_1 \leq \dots \leq k_w$, we say that vectors A and B from \mathbb{Z}^w are (k_1, \dots, k_w) -crossing when there are two coordinates i and j for which $A[i] - B[i] \geq k_i$ and $B[j] - A[j] \geq k_j$. Let $f(k_1, \dots, k_w; w)$ denote the maximum size of a (k_1, \dots, k_w) -crossing-free antichain of vectors in \mathbb{Z}^w . Thus $f(k, w) = f(k, \dots, k; w)$.

Proposition 13. *For $w \geq 1$ and $k_1, \dots, k_w \geq 1$, we have*

$$k_2 \cdots k_w \leq f(k_1, \dots, k_w; w) \leq k_1 \cdots k_w.$$

The proof of [Proposition 13](#) follows along the same lines as the proof of the inequalities $k^{w-1} \leq f(k, w) \leq k^w$ at the beginning of [Section 3](#). We propose a conjecture which seems to be more general but turns out to be equivalent to [Conjecture 1](#).

Conjecture 14. *For $w \geq 1$ and $1 \leq k_1 \leq \dots \leq k_w$, we have*

$$f(k_1, \dots, k_w; w) = k_2 \cdots k_w.$$

Proposition 15. *Conjectures 1 and 14 are equivalent.*

Proof. Clearly, [Conjecture 14](#) yields [Conjecture 1](#). To prove the converse implication, we assume $f(k_1, \dots, k_1; w) = k_1^{w-1}$ and prove $f(k_1, \dots, k_w; w) = k_2 \cdots k_w$. Let \mathcal{A} be a (k_1, \dots, k_w) -crossing-free antichain in \mathbb{Z}^w . For any selection of k_1 -element subsets $I_2 \subset \{0, \dots, k_2 - 1\}, \dots, I_w \subset \{0, \dots, k_w - 1\}$, consider the family $\mathcal{A}(I_2, \dots, I_w) = \{A \in \mathcal{A} : A[i] \bmod k_i \in I_i \text{ for } 2 \leq i \leq w\}$. Now, modify each $A \in \mathcal{A}(I_2, \dots, I_w)$ to get a vector A' so that if $A[j] = a_j k_j + r_j$, where $0 \leq r_j < k_j$, and ℓ_j is the position of r_j in the natural ordering of I_j , then $A'[j] = a_j k_1 + \ell_j$. Clearly, the resulting family \mathcal{A}' of all the vectors A' is a k_1 -crossing-free antichain. Thus $|\mathcal{A}(I_2, \dots, I_w)| = |\mathcal{A}'| \leq k_1^{w-1}$. Summing up over all selections of subsets I_2, \dots, I_w , we obtain

$$\binom{k_2-1}{k_1-1} \cdots \binom{k_w-1}{k_1-1} |\mathcal{A}| \leq \binom{k_2}{k_1} \cdots \binom{k_w}{k_1} k_1^{w-1},$$

which implies $|\mathcal{A}| \leq k_2 \cdots k_w$. \square

Proposition 15 tells us that in some sense the most difficult case is when all k_i are equal. Surprisingly, for some values of k_i , we know the exact answer. For instance,

$$f(k, k, 2k, \dots, 2^{w-1}k; w) = k \cdot 2k \cdots 2^{w-1}k.$$

Namely, we show that

$$f(k, k, 2k, \dots, 2^{w-1}k; w) \leq kf(2k, 2k, \dots, 2^{w-1}k; w - 1),$$

which together with $f(k, k; 2) = k$ and Proposition 13 gives the previous equality. We write $A <_1^2 B$ if $A[1] < B[1]$ and $A[2] > B[2]$. Every maximum chain in this order has size at most k , as otherwise it would yield a $(k, k, 2k, \dots, 2^{w-1}k)$ -crossing. Let \mathcal{A}' be a family of vectors of a fixed height in the order $<_1^2$. Now let $\phi(A) = (A[1] + A[2], A[3], \dots, A[w])$ for $A \in \mathcal{A}'$. The mapping ϕ is an injection, and $\phi(\mathcal{A}')$ is a $(2k, 2k, \dots, 2^{w-1}k)$ -crossing-free antichain in \mathbb{Z}^{w-1} . This gives the required inequality.

6. Extremal examples

Some classical extremal problems have elegant solutions due to the fact that all maximal structures are also maximum. For example, the maximum number of edges in a planar graph is $3n - 6$ when $n \geq 3$, because if G is any planar graph containing a face that is not a triangle, then an edge can be added to G while preserving planarity.

Other extremal problems can have many different maximal structures but essentially only one which is maximum. An example of this is Turán’s theorem, which asserts that the maximum number of edges in a graph on n vertices which does not contain a complete subgraph on $k + 1$ vertices is the number of edges in the complete k -partite graph on n vertices, where the part sizes are as balanced as possible. Another example is Sperner’s theorem, which asserts that the only maximum antichains in the lattice of all subsets of $\{1, \dots, n\}$ are the ranks at levels $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$.

It is our feeling that the extremal problem discussed in this paper is challenging because there are many different examples that we suspect to be extremal. We already presented one example at the beginning of Section 3, and in this section we develop some others.

6.1. Inductive construction

Suppose that we have constructed a k -crossing-free antichain \mathcal{A} in \mathbb{Z}^w , and suppose it is contained in $[0, c)^w$. We are going to construct an antichain \mathcal{A}' of size $k|\mathcal{A}|$ on $w + 1$

coordinates. Put k disjoint copies of \mathcal{A} one above another on coordinates $1, \dots, w$, that is, the i th copy inside $[(i - 1)c, ic]^w$, and set the coordinate $w + 1$ to be $-i$ for all vectors in the i th copy. This way we obtain a k times larger k -crossing-free antichain in \mathbb{Z}^{w+1} . If $|\mathcal{A}| = k^{w-1}$, then $|\mathcal{A}'| = k^w$.

6.2. Lexicographic construction

When $A \in \mathbb{Z}^w$, the rank of A is the sum $A[1] + \dots + A[w]$. Let $k, w \geq 2$. We construct an antichain \mathcal{A} in \mathbb{Z}^w as follows. First, consider the family \mathcal{F} of all vectors in \mathbb{Z}^w with $0 \leq A[i] \leq k - 1$ for $1 \leq i \leq w$ and $\sum_{i=1}^w A[i] \equiv w(k - 1) \pmod{k}$. Clearly, there are k^{w-1} vectors in \mathcal{F} . For each $A \in \mathcal{F}$, there is a unique non-negative integer $m(A)$ such that

$$m(A) \cdot k + A[1] + \dots + A[w] = w(k - 1).$$

Let n be the maximum value of $m(A)$ taken over all vectors $A \in \mathcal{F}$. Then, let $\tau = (i_1, \dots, i_n)$ be any sequence of integers from $\{1, \dots, w\}$. We modify \mathcal{F} into an antichain \mathcal{A} by the following rule. If $A \in \mathcal{F}$, then we modify A by increasing coordinate i by pk , where p is the number of times i occurs at the first $m(A)$ positions of τ . Clearly, these modifications result in a family \mathcal{A} consisting of k^{w-1} vectors. Furthermore, since each vector $A \in \mathcal{A}$ has rank $w(k - 1)$, we know that \mathcal{A} is an antichain. Also, \mathcal{A} is k -crossing-free.

The example presented at the beginning of Section 3 with the w th coordinate of all vectors shifted up by $w(k - 1)$ is the special case of this construction where τ is the constant sequence (w, \dots, w) .

6.3. Cyclic construction

Here, we fix $w = 3$ and consider coordinates $\{1, 2, 3\}$ in the cyclic order. Thus if $i = 3$ then $i + 1 = 1$, and if $i = 1$ then $i - 1 = 3$. Let $k \geq 2$. Consider the infinite family

$$\mathcal{F} = \{A \in \mathbb{Z}^3: A[i + 1] \leq A[i] + k \text{ and } A[i - 1] \leq A[i] + k - 1 \text{ for } i \in \{1, 2, 3\}\}.$$

Clearly, it contains no two k -crossing vectors. If $k \equiv 0$ or $k \equiv 2 \pmod{3}$, then the vectors in \mathcal{F} of rank $2k - 1$ form an antichain of size k^2 . If $k \equiv 1 \pmod{3}$, then the vectors in \mathcal{F} of rank $2k - 2$ form an antichain of size k^2 . In both cases, there is a cyclic symmetry between all three coordinates.

When $k \equiv 1 \pmod{3}$, the vectors in \mathcal{F} of rank $2k - 1$ form an antichain of size $k^2 - 1$ only. Still, we can add the vector $(\frac{k-1}{3} + k, \frac{k-1}{3}, \frac{k-1}{3})$ to obtain a k -crossing-free antichain of size k^2 at the price of losing the cyclic symmetry.

6.4. Remarks on rank

All the examples we have constructed so far are *ranked* antichains, that is, they consist of vectors in \mathbb{Z}^w all of which have the same rank. Based on this observation, it would be tempting to try to reduce the entire problem to ranked antichains. Indeed, we have the following proposition.

Proposition 16. *For all $k, w \geq 1$, the maximum size of a ranked k -crossing-free antichain in \mathbb{Z}^w is k^{w-1} .*

Proof. We only need to prove that if \mathcal{A} is a ranked k -crossing-free antichain in \mathbb{Z}^w , then $|\mathcal{A}| \leq k^{w-1}$. We can assume as before that $k, w \geq 2$. For each vector A in \mathcal{A} , let $\sigma(A)$ denote the vector in $\{0, \dots, k-1\}^{w-1}$ such that $A[i] \equiv \sigma(A)[i] \pmod{k}$ for $1 \leq i \leq w-1$. Clearly, σ is an injection and its range has at most k^{w-1} elements. \square

However, we know examples of k -crossing-free antichains in \mathbb{Z}^w of the conjectured extremal size k^{w-1} that are intrinsically non-ranked. For example, for $k = 2$ and $w = 4$, the following eight vectors form a non-ranked 2-crossing-free antichain in \mathbb{Z}^4 :

$$(0, 2, 1, 1), (2, 1, 0, 1), (1, 0, 2, 1), (1, 1, 1, 1), \\ (1, 3, 2, 0), (3, 2, 1, 0), (2, 1, 3, 0), (2, 2, 2, 0).$$

The first four of the vectors above have rank 4, while the last four have rank 6. Moreover, this antichain is compressed on each of the four coordinates. More generally, any family obtained by the cyclic construction (Section 6.3) can be extended to $w = 4$ in an analogous manner.

6.5. Remarks on the size of the largest coordinate

Proposition 17. *Let k and w be positive integers. Suppose that \mathcal{A} is a k -crossing-free antichain in \mathbb{Z}^w , and suppose that there is a coordinate j on which all vectors are different. Then $|\mathcal{A}| \leq k^{w-1}$.*

Proof. Suppose all vectors differ on the first coordinate. For $2 \leq i \leq w$, we define an order $<_i$ on \mathcal{A} as follows. We put $A <_i B$ if $A[1] < B[1]$ and $A[i] > B[i]$. The maximum size of a chain in this order is at most k , as otherwise \mathcal{A} would have two k -crossing vectors. Let $\phi(A) \in \{1, \dots, k\}^{w-1}$ be the vector of heights of A in orders $<_2, \dots, <_w$. Clearly, if $A, B \in \mathcal{A}$ are such that $A[1] < B[1]$, then for some coordinate i we have $A[i] > B[i]$, and thus the heights of A and B in $<_i$ are different. This shows that the mapping $\phi: \mathcal{A} \rightarrow \{1, \dots, k\}^{w-1}$ is injective. \square

It follows that in any k -crossing-free antichain in \mathbb{Z}^w the number of different values attained on any coordinate is at most k^{w-1} . Otherwise, a choice of representatives of the attained values would contradict [Proposition 17](#).

6.6. Remarks on compression

Careful analysis of the proof of the case $w = 3$ shows that we do not really need a fully compressed coordinate. We only use the following two properties:

- (P_1) For $1 \leq j \leq r$, the set $\mathcal{B}_j = \{A \in \mathcal{A} : A[3] = s + (j - 1)k\}$ is an antichain.
 (P_2) For $1 \leq j \leq r - 1$, there is a short path from a vector X_{j+1} in \mathcal{B}_{j+1} to a vector Y_j in \mathcal{B}_j .

However, when $w = 4$, this weaker notion of compression (with $A[3]$ replaced by $A[4]$) is not enough. To see this, consider the union of the following families of vectors in \mathbb{Z}^4 :

- (i) The vectors for which $0 \leq A[1], A[2] \leq k - 1, A[3] \geq 2, A[1] + A[2] + A[3] = 2k - 2$, and $A[4] = k$.
 (ii) All vectors of the form $(i, k - 1 - i, k + 1, 0)$ where $0 \leq i \leq k - 1$.
 (iii) The vector $(k - 1, k - 1, k, 0)$.
 (iv) All vectors having rank $3k - 2$ with $1 \leq A[i] \leq k - 1$ for $1 \leq i \leq 4$.

It is easy to see that this family satisfies properties (P_1) and (P_2) but has more than k^2 vectors for which $A[4] \equiv 0 \pmod{k}$.

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References

- [1] B. Bosek, T. Krawczyk, E. Szczyńska, First-fit algorithm for the on-line chain partitioning problem, *SIAM J. Discrete Math.* 23 (4) (2009/2010) 1992–1999.
 [2] R.P. Dilworth, A decomposition theorem for partially ordered sets, *Ann. of Math.* 51 (1) (1950) 161–166.
 [3] R.P. Dilworth, Some combinatorial problems on partially ordered sets, in: *Combinatorial Analysis*, in: *Proc. Sympos. Appl. Math.*, vol. 10, American Mathematical Society, Providence, RI, 1960, pp. 85–90.
 [4] V. Dujmović, G. Joret, D.R. Wood, An improved bound for first-fit on posets without two long incomparable chains, *SIAM J. Discrete Math.* 26 (3) (2012) 1068–1075.

- [5] S. Felsner, T. Krawczyk, W.T. Trotter, On-line dimension for posets excluding two long incomparable chains, *Order* 30 (1) (2013) 1–12.
- [6] P.C. Fishburn, Intransitive indifference with unequal indifference intervals, *J. Math. Psych.* 7 (1) (1970) 144–149.
- [7] G. Joret, K. Milans, First-fit is linear on posets excluding two long incomparable chains, *Order* 28 (3) (2011) 455–464.
- [8] K.M. Koh, On the lattice of maximum-sized antichains of a finite poset, *Algebra Universalis* 17 (1) (1983) 73–86.
- [9] B.G. Mirkin, Description of some relations on the set of real-line intervals, *J. Math. Psych.* 9 (2) (1972) 243–252.
- [10] E. Sperner, Ein Satz über Untermengen einer endlichen Menge, *Math. Z.* 27 (1) (1928) 544–548.