Discrete Mathematics 16 (1976) 361–381.

CHARACTERIZATION PROBLEMS FOR GRAPHS, PARTIALLY ORDERED SETS, LATTICES, AND FAMILIES OF SETS

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Received 22 September 1975
Revised 20 February 1976

A standard problem in combinatorial theory is to characterize structures which satisfy a certain property by providing a minimum list of forbidden substructures, for example, Kuratowski's well known characterization of planar graphs. In this paper, we establish connections between characterization problems for interval graphs, circular arc graphs, comparability graphs, planar lattices, 0–1 matrices, interval orders, partially ordered sets with dimension at most two, partially ordered sets with interval dimension at most two, and related problems involving the representation of families of finite sets by points or intervals of the real line. We use these connections to determine the collection of all 3-irreducible posets. We are then able to determine the collection of all 3-interval irreducible posets of height one and the set of all forbidden subgraphs with clique covering number two in the characterization of circular arc graphs.

1. Introduction

A frequently encountered problem in combinatorial theory is to characterize those structures which satisfy a certain property by providing a minimum list of "forbidden" substructures. In order for such a characterization to exist, it is necessary that all substructures of a structure satisfying the property also satisfy the property. In graph theory, the well known forbidden subgraph characterizations of planar graphs [22], outer planar graphs [6], and line graphs [2] illustrate this kind of characterization.

In general, characterization problems tend to be quite difficult as the list of forbidden substructures may be extensive; it may even involve infinite families. For example, consider the list of forbidden substructures for the characterizations of graphs of genus at most one and graphs with chromatic number at most three. Furthermore, it is often the case that a tedious series of ad hoc arguments is required to solve a characterization problem; as a consequence, the solution of the problem, while interesting in its own right, may fail to provide any significant advancement in our understanding of the structures or properties under investigation.

In this paper we will discuss a number of characterization problems involving interval graphs, comparability graphs, circular arc graphs, interval orders, planar lattices, the dimension of partially ordered sets, 0–1 matrices, and the representation of families of sets by points or intervals of the real line. Despite their apparent diversity these characterization problems are intimately related. We intend to explain in detail how progress towards the solution of any of the unsolved characterization problems discussed in this paper will contribute to our understanding of a wide range of combinatorial structures and properties.

2. Characterization problems

In this section, we briefly describe a number of characterization problems, pausing only to provide the essential definitions. In some cases, the connections between the problems are obvious, but where they are not, we defer the explanations to the last section of the paper, where we will proceed along historical lines, detailing the connections between the characterization problems introduced here.

In this paper, all graphs are finite and have no loops or multiple edges. We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$ respectively. A graph G is called the *intersection graph* of a family $\mathcal{F}$ of sets when there exists a bijection $f: V(G) \to \mathcal{F}$ so that $xy \in E(G)$ iff $f(x) \cap f(y) \neq \emptyset$. A graph G is called an *interval graph* when it is the intersection graph of a family of closed intervals (we adopt the convention of considering a point as a closed interval) of the real line E^1. Lekkerkerker and Boland [24] characterized interval graphs by providing the minimum collection $\mathcal{I}$ of graphs so that a graph G is an interval graph iff G does not contain a graph from $\mathcal{I}$ as an induced subgraph. It is convenient to group the graphs in $\mathcal{I}$ into three infinite families $\{C_n : n \geq 4\}$, $\{K_n : n \geq 1\}$, and $\{L_n : n \geq 1\}$ with two "odd" examples B_1 and B_2 left over. These graphs are shown in Fig. 1.

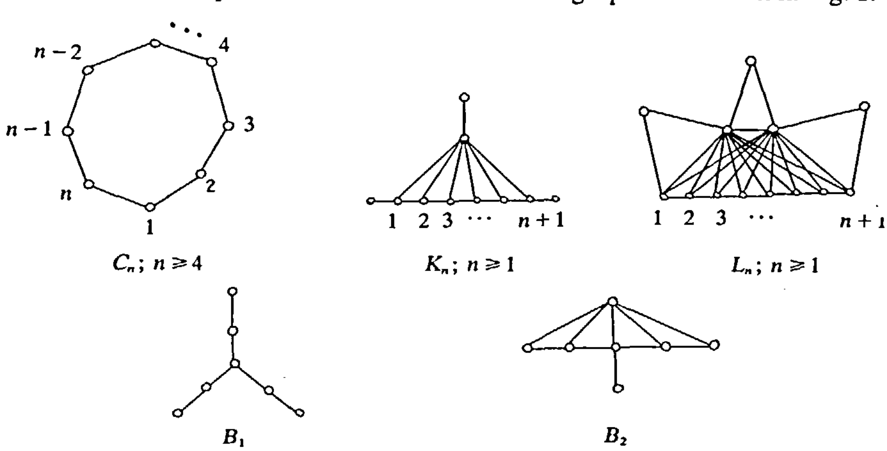


Fig. 1.

We consider partial orders as irreflexive and transitive binary relations and use the standard acronym poset for a partially ordered set. We also employ the standard notion of Hasse diagrams for posets.

A collection of closed intervals of E^1 has a natural partial order, called *domination* and denoted $\lhd$, defined by $A \lhd B$ iff $a < b$ in E^1 for all $a \in A$ and $b \in B$. A poset (X, P) is called an *interval order* if there exists a function f which assigns to each $x \in X$ a closed interval $f(x)$ of E^1 so that xPy iff $f(x) \lhd f(y)$ for all $x, y \in X$. Fishburn [9] determined the list of forbidden subposets which characterizes interval orders. The list consists of a single poset! It is the free sum of two 2-element chains (see Fig. 2).

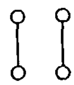
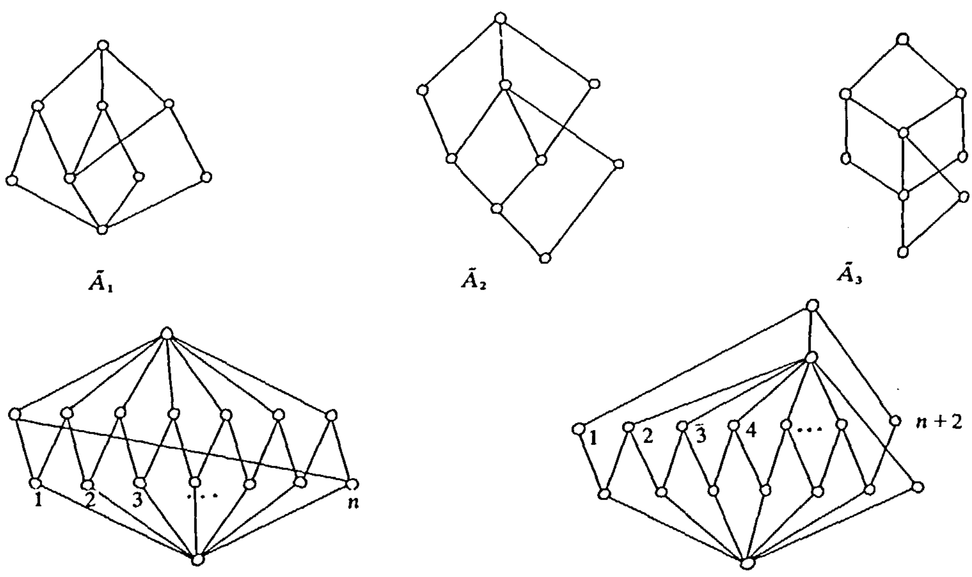


Fig. 2.

A poset (or lattice) is said to be *planar* if it has a planar Hasse diagram. Since a subposet of a planar poset need not be planar, there is no characterization of planar posets by a list of forbidden subposets. However, if a planar poset is also a lattice, then any subposet which is also a lattice is also planar. Therefore, it is possible to characterize planar lattices by providing the minimum list $\mathscr{L}$ of lattices so that a poset X which is also a lattice is planar iff it does not contain a lattice from $\mathscr{L}$ as a subposet. Kelly and Rival [16] have determined $\mathscr{L}$; in order to simplify the listing of the lattices in $\mathscr{L}$ it is customary not to include both a lattice and its dual when they are nonisomorphic. With this convention, $\mathscr{L}$ consists of three "odd" examples and five infinite families (see Fig. 3).



$\tilde{C}_n$; $n \geq 3$ Fig. 3. $\tilde{T}_n$; $n \geq 1$

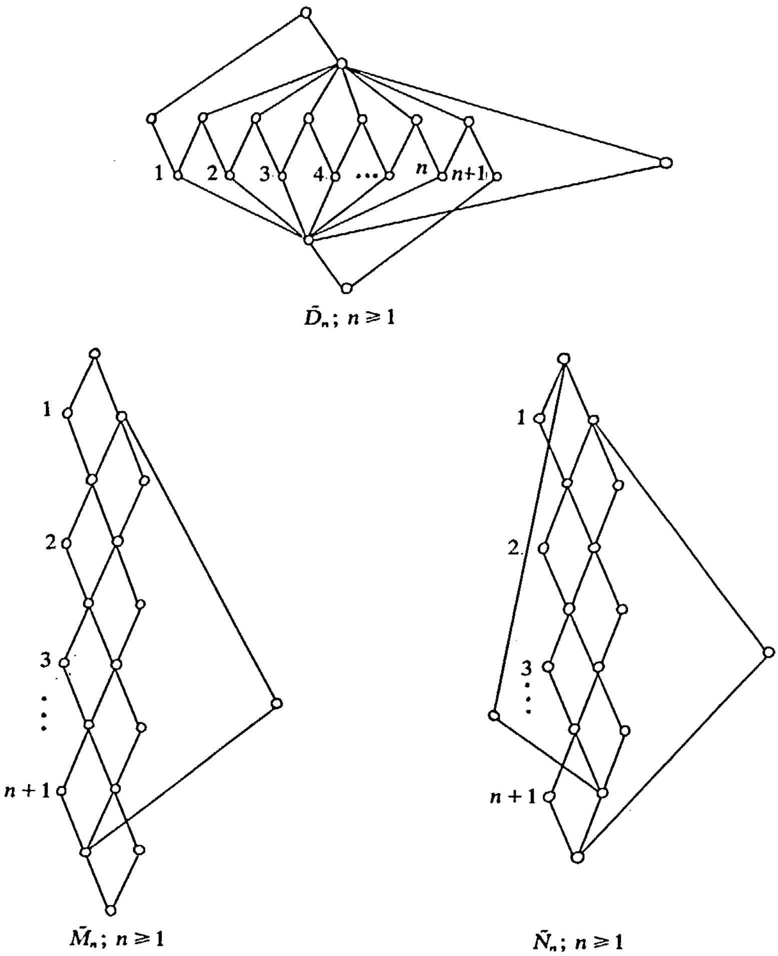


Fig. 3.

If S is an irreflexive binary relation on a set X, we may associate with S a graph $G(S)$ whose vertex set is X with x and y adjacent in $G(S)$ iff xSy or ySx. A graph G is called a *comparability graph* if there exists a partial order P so that G is isomorphic to $G(P)$. Equivalently, a graph is a comparability graph if it admits a transitive orientation of its edges. Using the results of Lekkerkerker and Boland [24] on interval graphs and the work of Gilmore and Hoffman [12] and Ghoula-Houri [11] on comparability graphs, Gallai [10] has determined the minimum collection $\mathscr{C}$ of graphs so that a graph is a comparability graph if and only if it does not contain an induced subgraph isomorphic to a graph in $\mathscr{C}$. The collection $\mathscr{C}$ consists of eight infinite families and ten odd examples. Following Gallai, we display the graphs in $\mathscr{C}$ in two parts. The graphs in Fig. 4(a) belong to $\mathscr{C}$ and the complements of the graphs in Fig. 4(b) belong to $\mathscr{C}$.

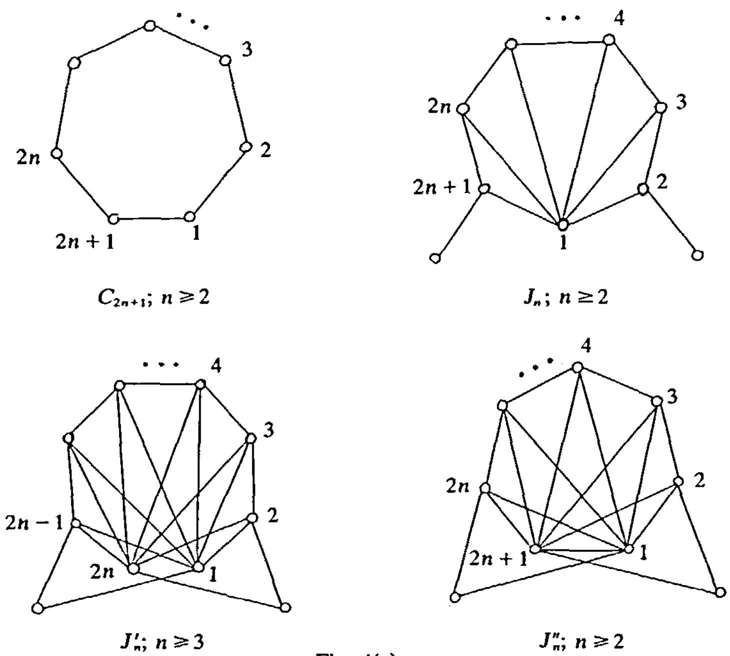

C_{2n+1}; $n \geq 2$
J_n; $n \geq 2$
J'_n; $n \geq 3$
J''_n; $n \geq 2$

Fig. 4(a).

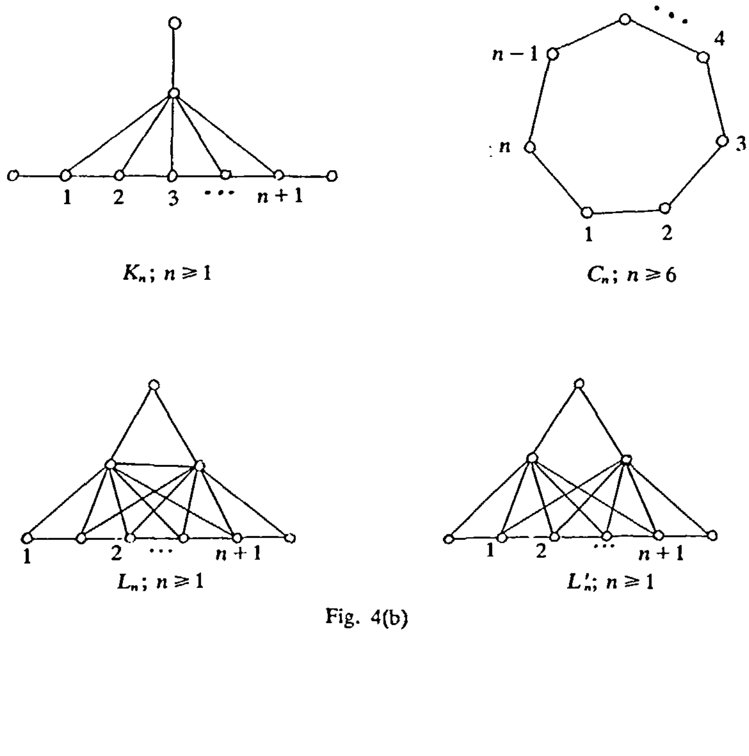

K_n; $n \geq 1$
C_n; $n \geq 6$
L_n; $n \geq 1$
L'_n; $n \geq 1$

Fig. 4(b)

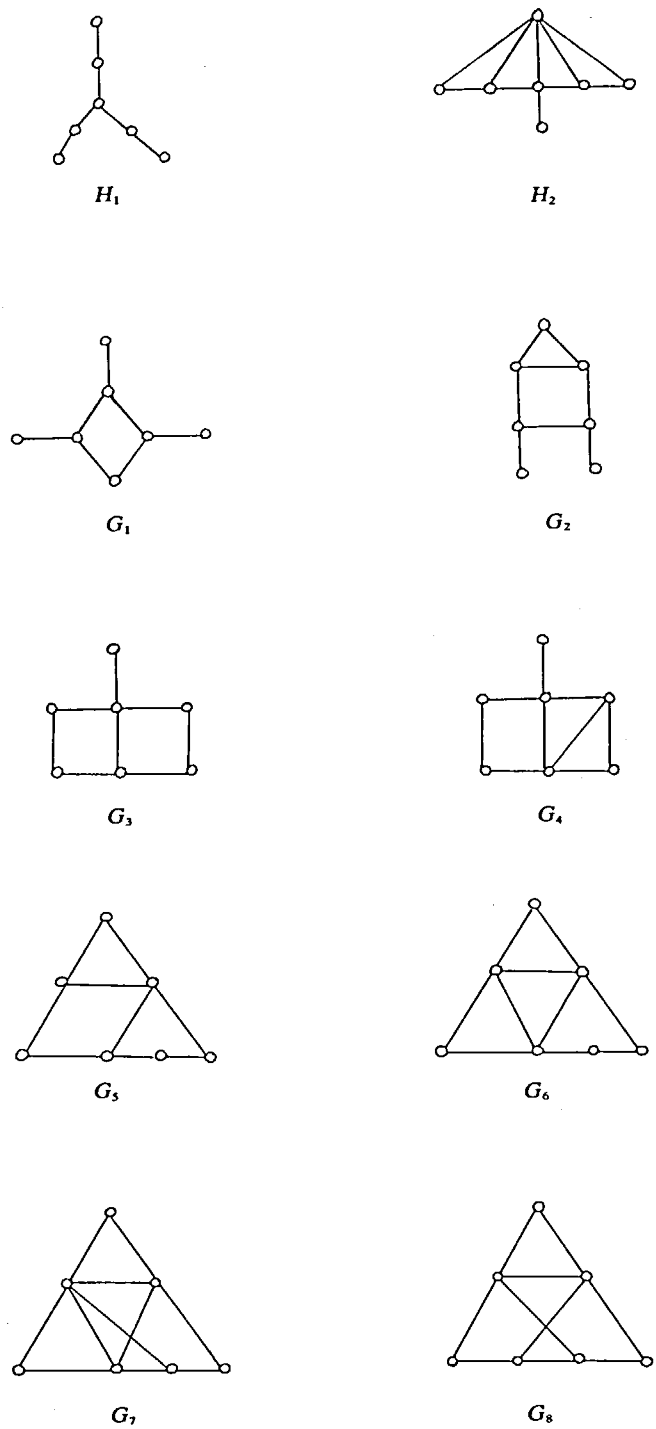


Fig. 4(b).

Let $\mathcal{F}$ and $\mathcal{G}$ be finite families of finite sets. We say $\mathcal{F}$ and $\mathcal{G}$ are *isomorphic* when there exists a bijection $f: \bigcup\mathcal{F} \to \bigcup\mathcal{G}$ so that $\mathcal{F} = \{f^{-1}(B): B \in \mathcal{G}\}$. In this paper we do not distinguish between isomorphic families. We say $\mathcal{F}$ is a derived subfamily of $\mathcal{G}$ when there exists an injection $g: \bigcup\mathcal{F} \to \bigcup\mathcal{G}$ so that $\mathcal{F} \subset \{g^{-1}(B): B \in \mathcal{G}\}$. For a finite set A belonging to a family $\mathcal{F}$ and a function $h: \bigcup\mathcal{F} \to \mathrm{E}^1$, we denote by $[h(A)]$ the smallest closed interval (possibly degenerate) of E^1 containing $\{h(x): x \in A\}$. We then say that a family $\mathcal{F}$ is *point-representable* if there exists a function $h: \bigcup\mathcal{F} \to \mathrm{E}^1$ so that $h(x) \in [h(A)]$ iff $x \in A$ for all $A \in \mathcal{F}$ and all $x \in A$ (see Eswaran [8] for an algorithm to determine if a family is point-representable).

Since a derived subfamily of a point-representable family is also point-representable, it is possible to characterize point-representability by determining the minimum list $\mathcal{R}$ of families so that a family $\mathcal{F}$ is point-representable iff it does not have a family from $\mathcal{R}$ as a derived subfamily. $\mathcal{R}$ contains three infinite families and two odd examples and is shown in Table 1. In the next section of the paper, we explain how $\mathcal{R}$ can be obtained from $\mathcal{I}$.

Table 1

$\mathcal{B}_1 = \{\{1,2,3\},\{1,4\},\{2,5\},\{3,6\}\}$,
$\mathcal{B}_2 = \{\{1,2\},\{3,4\},\{1,2,3,4\},\{2,3,5\}\}$,

$\mathcal{C}_3 = \{\{1,2\},\{2,3\},\{3,1\}\}$,
$\mathcal{C}_4 = \{\{1,2\},\{2,3\},\{3,4\},\{4,1\}\}$,
$\mathcal{C}_5 = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,1\}\}$,
$\vdots$

$\mathcal{K}_1 = \{\{1,2\},\{2,3\},\{2,4\}\}$,
$\mathcal{K}_2 = \{\{1,2\},\{2,3\},\{3,4\},\{2,3,5\}\}$,
$\mathcal{K}_3 = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{2,3,4,6\}\}$,
$\vdots$

$\mathcal{L}_1 = \{\{1,2\},\{2,3\},\{1,2,4\},\{2,3,4\}\}$,
$\mathcal{L}_2 = \{\{1,2\},\{2,3\},\{3,4\},\{1,2,3,5\},\{2,3,4,5\}\}$,
$\mathcal{L}_3 = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{1,2,3,4,6\},\{2,3,4,5,6\}\}$,
$\vdots$

We say a family $\mathcal{F}$ is *Δ-point-representable* if the family $\mathcal{F} \cup \{A - B: A, B \in \mathcal{F}\}$ is point-representable. Moore [25] determined the minimum list $\mathcal{R}^\Delta$ so that a family $\mathcal{F}$ is Δ-point-representable iff $\mathcal{F}$ does not have a family from $\mathcal{R}^\Delta$ as a derived subfamily. $\mathcal{R}^\Delta$ contains one infinite family $\{\mathcal{C}_n: n \geq 3\}$ as defined in Table 1 and the six "odd" families from Table 2.

We next discuss a natural generalization of these results which involves the representation of finite families of finite sets by intervals of the real line. If F assigns to each element a of A a closed interval $F(a)$ of E^1, we denote by $[F(A)]$ the smallest closed interval of E^1 containing each of the intervals $F(a)$ where $a \in A$.

Table 2

$\mathcal{F}_1 = \{\{1,2,3\},\{1\},\{2\},\{3\}\}$,
$\mathcal{F}_2 = \{\{1,2\},\{1,3\},\{1,4\}\}$,
$\mathcal{F}_3 = \{\{1,2,3\},\{1,2\},\{1,3\},\{1\}\}$,
$\mathcal{F}_4 = \{\{1,2,3,4\},\{1,2\},\{1,3\}\}$,
$\mathcal{F}_5 = \{\{1,2,3\},\{1,2\},\{1\},\{2\}\}$,
$\mathcal{F}_6 = \{\{1,2,3\},\{1,2,4\},\{1\}\}$.

We then say a family $\mathcal{F}$ is *interval-representable* if there exists a function F which assigns to each element x of $\bigcup\mathcal{F}$ a closed interval $F(x)$ of E^1 so that $F(x) \subset [F(A)]$ iff $x \in A$ for all $A \in \mathcal{F}$ and $x \in \bigcup\mathcal{F}$. We say that $\mathcal{F}$ is *Δ-interval-representable* if $\mathcal{F} \cup \{A - B : A, B \in \mathcal{F}\}$ is interval-representable. We then denote by $\mathcal{R}_1$ and $\mathcal{R}_1^\Delta$ the minimum lists of families which characterize interval-representability and Δ-interval-representability respectively. In the next section of this paper, we will determine both $\mathcal{R}_1$ and $\mathcal{R}_1^\Delta$.

A graph G is called a *circular arc graph* if it is the intersection graph of a family of arcs of a circle. Tucker [37, 38] provided a test for circular arc graphs involving the adjacency matrix (see also Klee [21]). Tucker [39] also gave a characterization theorem for proper circular arc graphs but the minimum list $\mathcal{A}$ of graphs so that a graph G is a circular arc graph iff it does not contain a graph from $\mathcal{A}$ as an induced subgraph has not been determined. However, we will determine the subcollection of $\mathcal{A}$ consisting of those graphs in $\mathcal{A}$ with clique covering number 2.

Dushnik and Miller [7] defined the *dimension* of a poset (X, P), denoted $\dim(X, P)$, as the smallest positive integer n for which there exist n linear orders $L_1, L_2, \ldots, L_n$ on X so that $P = L_1 \cap L_2 \cap \ldots \cap L_n$. Equivalently (see Ore [26]), $\dim(X, P)$ is the smallest positive integer n for which there exists a function f from X to n-dimensional Euclidean space E^n so that $x \leq y$ in P iff $f(x)(i) \leq f(y)(i)$ in E^1 for $1 \leq i \leq n$. We will sometimes abuse notation and use a single symbol (usually X) to denote a poset. In this case, we will write $\dim(X)$ for the dimension of the poset X; we will also denote the comparability graph of the poset X by $G(X)$. If $\dim(X) = n \geq 2$ and $\dim(X - x) < n$ for all $x \in X$, we say that X is *n-irreducible.* The only 2-irreducible poset is a two-element antichain.

A poset has dimension one iff it is a chain (linear order). Dushnik and Miller [7] provided a test for posets with dimension at most two: the dimension of a poset is at most two iff it is isomorphic to a collection of closed intervals of the real line ordered by inclusion. This test was generalized by Leclerc [23] and Trotter and Moore [35]. In this paper, we will determine the minimum list $\mathcal{P}$ of posets so that a poset X has dimension at most two iff it does not contain a poset from $\mathcal{P}$ as a subposet. $\mathcal{P}$ is the collection of all 3-irreducible posets.

Trotter and Bogart [32] introduced the concept of the *interval* dimension of a poset (X, P), denoted $\mathrm{Idim}(X, P)$, as the smallest positive integer n for which there exists a function F which assigns to each $x \in X$ a sequence $\{F(x)(i) \colon 1 \leq i \leq n\}$ of closed intervals of E^1 so that xPy iff $F(x)(i) \lhd F(y)(i)$ for $1 \leq i \leq n$. As before, we will write Idim(X) for the interval dimension of the poset X. It is clear that $\mathrm{Idim}(X) \leq \dim(X)$ for all X.

If $\mathrm{Idim}(X) = n \geq 2$ and $\mathrm{Idim}(X - x) < n$ for all $x \in X$, we say that X is *n-interval-irreducible.* A poset X has interval dimension one iff it is an interval order; the only 2-interval-irreducible poset is shown in Fig. 2.

We denote by $\mathcal{P}_I$ the minimum list of posets so that a poset has interval dimension at most two iff it does not contain a poset from $\mathcal{P}_I$ as a subposet. $\mathcal{P}_I$ is the collection of all 3-interval-irreducible posets. In the next section, we will determine the subcollection of $\mathcal{P}_I$ consisting of those posets of height one which are 3-interval-irreducible.

3. Combinatorial connections

In the preceding sections, we have described a number of characterization problems but have made no attempt to motivate these problems or to explain any combinatorial connections between them. In this section, we intend to justify our claim that these apparently diverse characterization problems are, in fact, intimately related. Because the authors' research efforts over the last few years have been concentrated on posets, we shall use these structures as a starting point.

In [27] Trotter defined the *crown* S_n^k for $n \geq 3$ and $k \geq 0$ as the poset of height one with maximal elements $a_1, a_2, \ldots, a_{n+k}$ and minimal elements $b_1, b_2, \ldots, b_{n+k}$. The partial order on S_n^k is defined by letting each b_i be incomparable with $a_i, a_{i+1}, \ldots, a_{i+k}$ (cyclically) and less than the remaining $n-1$ maximal elements. The crown S_n^0 is isomorphic to the poset consisting of all one-element and $(n-1)$-element subsets of an n-element set ordered by inclusion; Hiraguchi [14] noted that S_n^0 is n-irreducible. Baker, et al. [1] and Kelly and Rival [18] studied the family of crowns $\{S_3^k \colon k \geq 0\}$; these posets are all 3-irreducible. Trotter's formula giving the dimension of the crown S_n^k is $\dim(S_n^k) = \{2(n+k)/(k+2)\}$.

One of the best known results in dimension theory for posets is Hiraguchi's inequality (see [4, 13, 14, 29]), $\dim(X) \leq \frac{1}{2}|X|$ when $|X| \geq 4$ (see [28] for a simple proof). Bogart and Trotter [5] gave a forbidden subposet characterization of Hiraguchi's inequality by determining for each $n \geq 2$, the minimum collection $\mathcal{H}_n$ of posets so that if $|X| \leq 2n$, then $\dim(X) < n$ unless X contains a poset from $\mathcal{H}_n$ as a subposet. The determination of $\mathcal{H}_2$ is trivial; it contains only a two-element antichain. $\mathcal{H}_3$ contains the 3-irreducible crown S_3^0, the "chevron" shown in Fig. 5, and its dual.

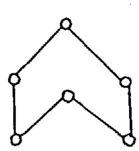


Fig. 5.

For $n \geq 4$, $\mathcal{H}_n$ contains a single poset, the crown S_n^0.

Trotter[1] extended this characterization by determining for each $n \geq 2$, the minimum list $\mathcal{H}'_n$ of posets so that if $|X| \leq 2n+1$, then $\dim(X) < n$ unless X contains a poset from $\mathcal{H}'_n$. If $n > 3$, $\mathcal{H}_n = \mathcal{H}'_n$, but $\mathcal{H}'_3$ contains no less than twenty-three posets. $\mathcal{H}'_3$ contains the posets which belong to $\mathcal{H}_3$ as well as the posets in Fig. 6. As in Section 2, we adopt the convention of not drawing both a poset and its dual when they are nonisomorphic.

Most of the difficulty encountered in proving that $\mathcal{H}'_n = \{S_n^0\}$ when $n \geq 4$ occurs when $n = 4$. The problem is to show that, while there are twenty 3-irreducible posets on 7 points, there are no 4-irreducible posets on 9 points.

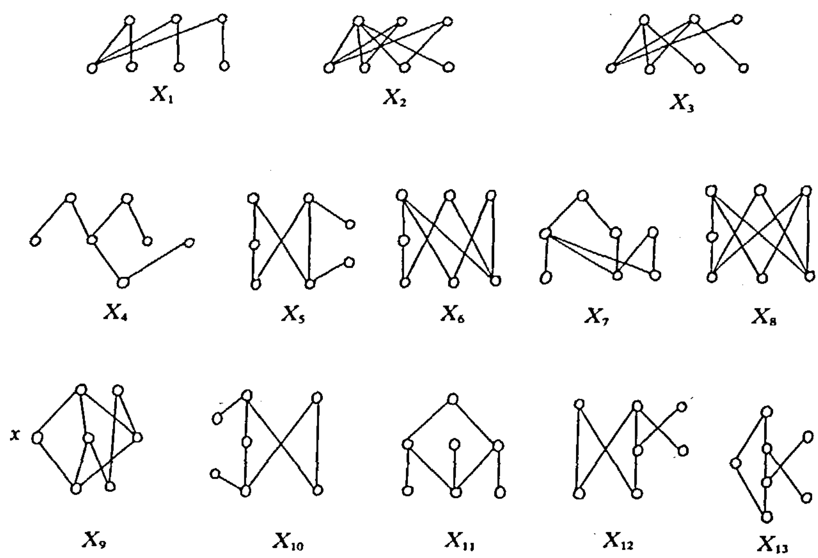


Fig. 6.

The existence of a large number of "odd" examples of 3-irreducible posets suggested quite naturally the problem of determining $\mathcal{P}$, the collection of all 3-irreducible posets. The fact that $\mathcal{H}'_n = \{S_n^0\}$ for $n \geq 4$ suggested strongly that the posets in $\mathcal{H}'_3$ were pathological examples of the type frequently encountered in the first case of combinatorial problems. This conjecture was supported by the fact that some of the posets in Fig. 6 also occur as pathology in another characterization problem [30].

The determination of $\mathcal{H}'_3$ was simplified by the following practical test for dimension at most two. For each point $x \in X$, choose a point p_x in the plane so that $x < y$ in X iff p_y is above and to the right of p_x. If a satisfactory choice can be made

[1] Kimble [19] independently established this characterization theorem in a much more compact fashion. Kimble's proof contains some interesting lemmas which have been applied to other characterization problems (see [30, 33]). His research on this problem also produced the notion of the split of a poset, a concept of great importance.

for each point in X, we conclude by Ore's definition of dimension that $\dim(X) \leqslant 2$. The choice of points in the plane is simplified by drawing horizontal and vertical rays in the positive directions from each point chosen. For example the "quarter-planes" in Fig. 7(b) show that the poset in Fig. 7(a) has dimension at most two.

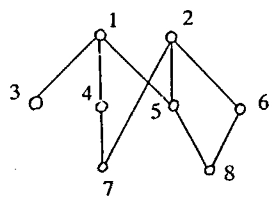


Fig. 7(a).

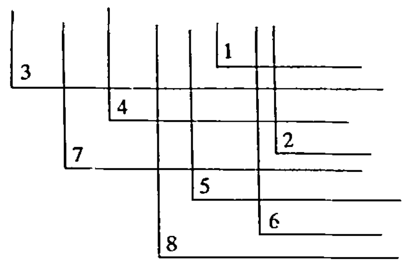


Fig. 7(b).

When $\dim(X) \geqslant 3$, an elementary argument to establish this fact can often be constructed using the geometric implications of these quarter-plane drawings. First we note that an antichain appears as a "saw-tooth" configuration in a drawing of a poset with dimension at most two; e.g., consider the antichain $\{3, 4, 5, 6\}$ in Fig. 7(b). If A is an antichain, $b \not\in A, b$ is comparable with the points in $A' \subset A$ and incomparable with the points in $A - A'$, then the corners corresponding to points in A' must be "together" in the saw-tooth configuration. For example, consider the antichain $A = \{3, 4, 5, 6\}$ and the point $2 \not\in A$. The point 2 requires that the corners for 5 and 6 be together in Fig. 7(b).

If A, A' and b have been chosen as in the preceding paragraph, and we choose $c \not\in A$ so that c is comparable with all points in $A'' \subset A'$ but incomparable with b and all points in $A - A''$, then the corners corresponding to points in $A' - A''$ must also be together. In Fig. 8, we show two subposets which would require that the corners corresponding to 1 and 3 in the antichain $\{1, 2, 3\}$ be together. Note that the duals of these subposets will also force 1 and 3 to be together.

Fig. 8.

In Fig. 9 we show three 3-irreducible posets. The first of these is S_3^0. The dimension of S_3^0 must clearly be at least three for if $\dim(S_3^0) \leqslant 2$, then each pair of elements in the antichain $\{a_1, a_2, a_3\}$ would necessarily be together in a quarter-plane drawing, which is clearly impossible. The second poset in Fig. 9 is the chevron of Fig. 5; it is obtained by flipping a minimal element in S_3^0 above the antichain of maximal elements so that the geometric force it imposes on a_1, a_2, a_3 in a quarter-plane drawing is preserved. The third poset is obtained in a similar fashion from the second. It is the poset X_7 in Fig. 6.

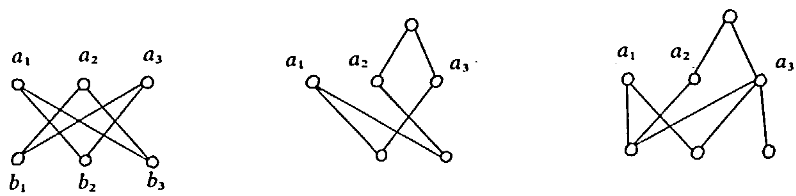


Fig. 9.

It is straightforward to construct all of the posets in $\mathscr{H}_3'$ in this fashion. We note however that while dimension at least three is preserved, irreducibility may not be preserved.

The quarter-plane drawings used to test a poset for dimension at most two provided a method for determining the collection of all 3-irreducible posets of height one. Given a poset X of height one, label the maximal elements $a_1, a_2, \ldots, a_m$ and the nonmaximal elements $b_1, b_2, \ldots, b_n$. Then define a family $\mathscr{F}_X = \{A_i: 1 \leq i \leq n\}$ of subsets of $\{1, 2, 3, \ldots, m\}$ by $j \in A_i$ iff $b_i < a_j$ in X. Consideration of the saw-tooth configuration determined by the corners corresponding to elements of the antichain $A = \{a_1, a_2, \ldots, a_m\}$ in a quarter-plane drawing of X shows immediately that $\dim(X) \leq 2$ iff $\mathscr{F}_X$ is Δ-point-representable. Furthermore, deleting a maximal element of X corresponds to restricting $\mathscr{F}_X$ to a proper subset of $\bigcup \mathscr{F}_X$ while deleting a nonmaximal element of X corresponds to deleting a set from $\mathscr{F}_X$. Therefore we see that X is a 3-irreducible poset of height one iff $\mathscr{F}_X \in \mathscr{R}^\Delta$.

Theorem 1. *If X is a poset of height one, then $X \in \mathscr{P}$ iff $\mathscr{F}_X \in \mathscr{R}^\Delta$.*

The solution of the characterization problem for Hiraguchi's inequality provided Moore with the correct conjecture for the posets in $\mathscr{R}^\Delta$ (see Table 2). From this list we conclude that the only 3-irreducible posets of height one are the crowns $\{S_3^k: k \geq 0\}$ and the first three posets (and their duals) of Fig. 6. We have listed the six families in Table 2 so that $\mathscr{F}_{2i-1}$ and $\mathscr{F}_{2i}$ are dual, and $\mathscr{F}_{2i-1} = \mathscr{F}_{X_i}$ for $i = 1, 2, 3$. We note, however, that there is no notion of duality for point-respresentability.

Moore's determination of $\mathscr{R}^\Delta$ was preceded by the determination of $\mathscr{R}$. Eswaran [8] gave an algorithm for determining whether a family is point-representable but did not establish the immediate connection between point-representable families and interval graphs. We will now sketch a simple method used by Trotter for determining $\mathscr{R}$.

For a family $\mathscr{F}$, we denote the *augmented* family $\mathscr{F} \cup \{\{x\}: x \in \bigcup \mathscr{F}\}$ by $\mathscr{F}^*$. The following theorems involving $\mathscr{F}^*$ are elementary.

Theorem 2. *$\mathcal{F}$ is point-representable iff the intersection graph of $\mathcal{F}^*$ is an interval graph.*

Theorem 3. *If the intersection graph of $\mathcal{F}^*$ contains the graph L_1 from Fig. 1, then $\mathcal{C}_3 = \{\{1,2\},\{2,3\},\{3,1\}\}$ is a derived subfamily of $\mathcal{F}$.*

Eswaran [8] observed that if $\mathcal{F}$ is point-representable and every pair of sets from $\mathcal{F}$ has nonempty intersection, then $\bigcap \mathcal{F} \neq \emptyset$. For each graph G in $\mathcal{I} - \{L_1\}$, label the nonsimplicial vertices $v_1, v_2, \ldots, v_m$ and the cliques $H_1, H_2, \ldots, H_n$. Then construct a family $\mathcal{F}_G = \{A_1, A_2, \ldots, A_m\}$ of subsets of $\{1,2,3,\ldots,n\}$ by the rule $j \in A_i$ iff v_i is a vertex in H_j. In view of Theorems 2, 3, and Eswaran's observation, it is easy to see that if $G \in \mathcal{I} - \{L_1\}$ and G is an induced subgraph of the intersection graph of $\mathcal{F}^*$, then $\mathcal{F}_G$ is a derived subfamily of $\mathcal{F}$. Therefore each family $\mathcal{F} \in \mathcal{R}$ with $\mathcal{F} \neq \mathcal{C}_3$ is of the form $\mathcal{F} = \mathcal{F}_G$ for some $G \in \mathcal{I}$. It is trivial to verify that the converse also holds. We conclude that $\mathcal{R} = \{\mathcal{F}_G : G \in \mathcal{I},\ G \neq L_1\} \cup \{\mathcal{C}_3\}$.

Alternately, $\mathcal{R}$ can be determined from the work of Tucker [40] on 0–1 matrices. Tucker defined a 0–1 matrix to have the *consecutive ones property* when it is possible to permute the rows so that the ones in each column appear consecutively. If a matrix M has the consecutive ones property, then any submatrix obtained from M by deleting a row or a column also has the consecutive ones property. Tucker determined the minimum list $\mathcal{M}$ of 0–1 matrices so that a matrix has the consecutive ones property iff it does not contain a matrix from $\mathcal{M}$ as a submatrix.

With an $m \times n$ 0–1 matrix M, we may associate a family $\mathcal{F}_M = \{A_j : 1 \leq j \leq n\}$ of subsets of $\{1,2,3,\ldots,m\}$ defined by $i \in A_j$ iff $m_{ij} = 1$. It is easy to see that M has the consecutive ones property iff $\mathcal{F}_M$ is point-representable. Therefore $\mathcal{R} = \{\mathcal{F}_M : M \in \mathcal{M}\}$. It should be noted that Tucker did not use the Lekkerkerker–Boland characterization of interval graphs to determine $\mathcal{M}$ although his development proceeds along parallel lines.

In 1972 Kimble [20] suggested a powerful method for converting an arbitrary dimension theory problem to a problem involving only posets of height one. First he defined the *split of a point x in a poset X*, denoted $S(x, X)$, as the poset obtained by adding to the subposet $X - x$ a maximal element x' and a minimal element x'' with $x' > x''$, $x' > y$ in $S(x, X)$ iff $x > y$ in X, and $y > x''$ in $S(x, X)$ iff $y > x$ in X. It is easy to establish the following result.

Theorem 4 (Kimble [20]). *For every poset X and every $x \in X$,*

$$\dim(X) \leq \dim S(x, X) \leq 1 + \dim(X).$$

Kimble applied this technique to construct a 3-irreducible poset on 8 points. He split the point labeled x in X_9 as shown in Fig. 6 to obtain the poset in Fig. 10.

Kimble also defined the *split of a poset X*, which we denote $S(X)$. It is the poset (of height one) obtained from X by splitting consecutively each point in X exactly

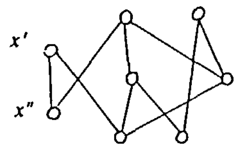


Fig. 10.

once. If X is a poset and C is a chain containing $2|X|$ points, then Kimble noted that $S(X)$ is a subposet of the cartesian product $X \times C$. We have then the following result.

Theorem 5 (Kimble [20]). *For every poset* X, $\dim(X) \leq \dim S(X) \leq 1 + \dim(X)$.

We pause to mention that Trotter and Moore [34] have proved that repeated splitting of a poset can increase the dimension at most twice; however it is an open problem to determine conditions on X which guarantee that $\dim S(X) = 1 + \dim(X)$.

As stated previously, Kelly and Rival [16] determined the collection $\mathscr{L}$ of lattices which characterize planar lattices. It is an easy exercise in Birkhoff [3] to show that a lattice is planar iff its dimension as a poset is at most two. Therefore all of the lattices in $\mathscr{L}$ are three dimensional! Furthermore for each lattice $L \in \mathscr{L}$, it is easy to extract a 3-irreducible subposet of L. Simply removing the universal bounds from the first three infinite families in Fig. 3 produces 3-irreducible posets which we will denote by $\{C_n : n \geq 3\}$, $\{T_n : n \geq 1\}$ and $\{D_n : n \geq 1\}$ respectively. Note that $C_i = S_3^{i-3}$ for all $i \geq 3$. Kelly and Rival [17] showed that removing the universal bounds and the doubly reducible elements from the last two infinite families produces 3-irreducible posets which we denote by $\{M_n : n \geq 1\}$ and $\{N_n : n \geq 1\}$ respectively. These families are shown in Fig. 11. Note that M_1 and N_1 appear in Fig. 6 as X_9 and X_{10} respectively.

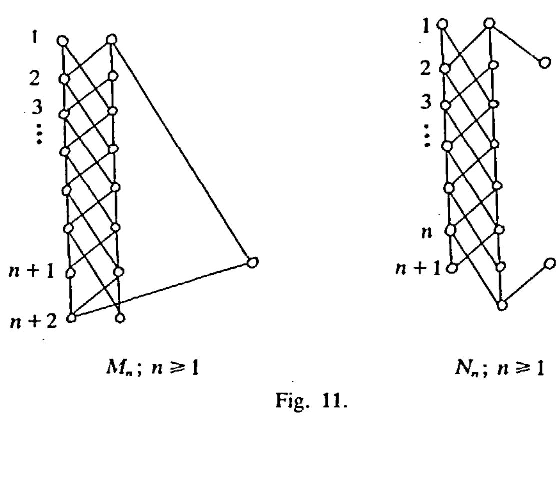


Fig. 11.

In view of Kimble's example (Fig. 10), Trotter suggested that splitting the point x in each poset in the first family in Fig. 11 might produce another family of 3-irreducible posets; this conjecture may be easily verified using quarter-plane arguments (Fig. 12).

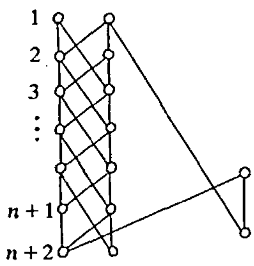


Fig. 12.

We define the *vertical*[2] *split of a point*, denoted $\mathrm{VS}(x, X)$, in a poset X as the poset obtained from X by adding a point x'' to X with $x > x''$ in $\mathrm{VS}(x, X)$, $x'' > y$ in $\mathrm{VS}(x, X)$ iff $x > y$ in X, and $y > x''$ in $\mathrm{VS}(x, X)$ iff $y > x$. Similarly we define the *vertical split of a poset* X, $\mathrm{VS}(X)$, as the poset obtained from X by vertically splitting each point in X exactly one time. It is then trivial to establish the following result which gives an immediate connection between dimension, interval dimension, and splits.

Theorem 6.

$$\dim(X) = \dim \mathrm{VS}(X),$$

$$\dim(X) = \mathrm{Idim}\, \mathrm{S}(X) = \mathrm{Idim}\, \mathrm{VS}(X).$$

At this point we establish the connection with the concepts of interval dimension, interval representability, and circular arc graphs. These connections are based on a test for interval dimension at most two. Suppose X is a poset with interval dimension at most two and let F be a function which assigns to each $x \in X$ a pair $F(x)(1)$, $F(x)(2)$ of closed intervals of the real line so that $x < y$ in X iff $F(x)(1) \lhd F(y)(1)$ and $F(x)(2) \lhd F(y)(2)$. Without loss of generality we may assume $1 \lhd F(x)(i) \lhd 2$ for all $x \in X$ and $i = 1, 2$. Now suppose further that X is a poset of height one. We produce a new interval representation of X by assigning to each minimal element b the intervals $G(b)(i) = [0, \max F(b)(i)]$ and to each maximal element a the interval $G(a)(i) = [\min F(a)(i), 3]$. Now consider the lines l_1 and l_2 in the plane through $(1, 1)$ and $(2, 2)$ respectively with slope -1. For each maximal element a, the rectangle as determined by G intersects l_2 in a closed

[2] It is natural to view the Kimble split of a poset as a "horizontal" split.

interval which we denote $G(a)$. Similarly a rectangle for a minimal element b intersects l_1 in a closed interval $G(b)$. Note that each interval $G(a)$ contains $(2,2)$ and each $G(b)$ contains $(1,1)$. If we construct the family $\mathcal{F}_X$ as defined earlier in this section, we then see clearly that $\mathcal{F}_X$ is interval-representable. As before we may conclude the following result.

Theorem 7. *If X is a poset of height one, then X is 3-interval irreducible iff $\mathcal{F}_X \in \mathcal{R}_1$.*

We now turn our attention to the list $\mathcal{A}$ of forbidden graphs in the characterization of circular arc graphs. Some information about $\mathcal{A}$ (including a listing of all the disconnected graphs in $\mathcal{A}$) can be obtained from Lekkerkerker and Boland's list of $\mathcal{I}$, the list of forbidden graphs in the characterization of interval graphs. If $G \in \mathcal{I}$ and G is not a circular arc graph, then $G \in \mathcal{A}$. On the other hand if $G \in \mathcal{I}$ and G is a circular arc graph, then the free sum of G and a trivial graph is in $\mathcal{A}$. Furthermore every disconnected graph in $\mathcal{A}$ is obtained in this manner; thus we may restrict our attention to those connected graphs in $\mathcal{A}$ which are not in $\mathcal{I}$. Some of these graphs are shown in Fig. 13.

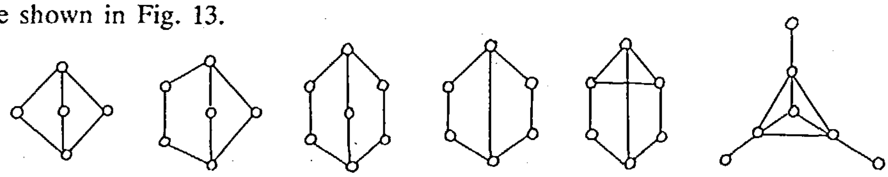


Fig. 13.

The *clique covering number* of a graph G, denoted $\mathrm{CCN}(G)$, is the smallest positive integer n for which the vertex set of G can be partitioned into n subsets so that each subset induces a complete subgraph of G. In what follows we will be concerned primarily with graphs with clique covering number two. We denote by $\mathcal{A}_2$ the set $\{G \in \mathcal{A} : \mathrm{CCN}(G) = 2\}$. We also denote by $\bar{G}$ the complement of graph G and for a poset X, we denote by $\bar{G}(X)$ the complement of the comparability graph of X. Now suppose X is a poset of height one for which $\mathrm{Idim}(X) \leq 2$. Then $\mathcal{F}_X$ is interval representable so we may choose a representation of $\mathcal{F}_X$ by intervals (arcs) on the upper semicircle of the unit circle. By using the representation described earlier in this section we may further assume that all of the arcs chosen contain the point $(0,1)$. Now to each minimal element b in X we associate an arc containing $(0,-1)$ which intersects each of the arcs corresponding to a maximal element a for which $b \nless a$. The collection of arcs determined by this process shows that the complement of the comparability graph of X is a circular arc graph. The following result follows immediately.

Theorem 8. *Let X be a poset of height one. Then the following statements are equivalent.*

(a) $X \in \mathcal{P}_1$,

(b) $\mathcal{F}_X \in \mathcal{R}_1$,

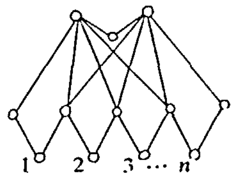
(c) $\bar{G}(X) \in \mathscr{A}_2$.

3-interval irreducible posets of height one can be obtained by taking the Kimble split of the 3-irreducible posets presented earlier and then extracting 3-interval irreducible posets. Some pathology is encountered in the posets of small order but the process smoothes out for the larger examples. Other families can be obtained by starting with the families in $\mathscr{R}$ and asking whether they are interval representable. For example, it is easy to see that although each $\mathscr{K}_n$ is interval representable, $\mathscr{K}_n \cup \{\{n+3\}\}$ is not; similarly each $\mathscr{L}_n$ is interval representable but $\mathscr{L}_n \cup \{\{n+3\}\}$ is not. In Table 3 we list all families that may be obtained in this fashion. In view of Theorems 7 and 8, duality is present in interval representation so we do not include both a family and its dual when they are nonisomorphic.

Table 3

$$
\begin{aligned}
&\mathscr{C}_3 = \{\{1,2\},\{2,3\},\{3,1\}\},\\
&\mathscr{C}_4 = \{\{1,2\},\{2,3\},\{3,4\},\{4,1\}\},\\
&\mathscr{C}_5 = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,1\}\},\\
&\quad\vdots\\[1ex]
&\mathscr{T}_1 = \{\{1,2\},\{2,3\},\{3,4\},\{2,3,5\},\{5\}\},\\
&\mathscr{T}_2 = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{2,3,4,6\},\{6\}\},\\
&\mathscr{T}_3 = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{2,3,4,5,7\},\{7\}\},\\
&\quad\vdots\\[1ex]
&\mathscr{W}_1 = \{\{1,2\},\{2,3\},\{1,2,4\},\{2,3,4\},\{4\}\},\\
&\mathscr{W}_2 = \{\{1,2\},\{2,3\},\{3,4\},\{1,2,3,5\},\{2,3,4,5\},\{5\}\},\\
&\mathscr{W}_3 = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{1,2,3,4,6\},\{2,3,4,5,6\},\{6\}\},\\
&\quad\vdots\\[1ex]
&\mathscr{D}_1 = \{\{1,2,5\},\{2,3,5\},\{3\},\{4,5\},\{2,3,4,5\}\},\\
&\mathscr{D}_2 = \{\{1,2,6\},\{2,8,6\},\{3,4,6\},\{4\},\{5,6\},\{2,3,4,5,6\}\},\\
&\mathscr{D}_3 = \{\{1,2,7\},\{2,3,7\},\{3,4,7\},\{4,5,7\},\{5\},\{6,7\},\{2,3,4,5,6,7\}\},\\
&\quad\vdots\\[1ex]
&\mathscr{M}_1 = \{\{1,2,3,4,5\},\{1,2,3\},\{1\},\{1,2,4,6\},\{2,4\},\{2,5\}\},\\
&\mathscr{M}_2 = \{\{1,2,3,4,5,6,7\},\{1,2,3,4,5\},\{1,2,3\},\{1\},\{1,2,3,4,6,8\},\\
&\qquad \{1,2,4,6\},\{2,4\},\{2,7\}\},\\
&\mathscr{M}_3 = \{\{1,2,3,4,5,6,7,8,9\},\{1,2,3,4,5,6,7\},\{1,2,3,4,5\},\{1,2,3\},\{1\},\\
&\qquad \{1,2,3,4,5,6,8,10\},\{1,2,3,4,6,8\},\{1,2,4,6\},\{2,4\},\{2,9\}\},\\
&\quad\vdots\\[1ex]
&\mathscr{N}_1 = \{\{1,2,3\},\{1\},\{1,2,4,6\},\{2,4\},\{2,5\},\{6\}\},\\
&\mathscr{N}_2 = \{\{1,2,3,4,5\},\{1,2,3\},\{1\},\{1,2,3,4,6,8\},\{1,2,4,6\},\{2,4\},\{2,7\},\{8\}\},\\
&\mathscr{N}_3 = \{\{1,2,3,4,5,6,7\},\{1,2,3,4,5\},\{1,2,3\},\{1\},\{1,2,3,4,5,6,8,10\},\\
&\qquad \{1,2,3,4,6,8\},\{1,2,4,6\},\{2,4\},\{2,9\},\{10\}\},\\
&\quad\vdots\\[1ex]
&\mathscr{G}_1 = \{\{1,3,5\},\{1,2\},\{3,4\},\{5,6\}\},\\
&\mathscr{G}_2 = \{\{1\},\{1,2,3,4\},\{2,4,5\},\{2,3,6\}\},\\
&\mathscr{G}_3 = \{\{1,2\},\{3,4\},\{5\},\{1,2,3\},\{1,3,5\}\}.
\end{aligned}
$$

We may use the families compiled in Table 3 to check for new 3-irreducible posets by asking if $S(X)$ contains a poset Y for which $\mathcal{F}_Y$ is given in Table 3, then what can we say about X. Again, some pathology is encountered when $|X|$ is small, but in general there is a great deal of regularity involved and it is straightforward (although tedious) to associate the 3-irreducible posets described earlier with the proper families in Table 3. However, one new family of 3-irreducible posets is obtained in the process. This family is associated with $\{\mathcal{W}_n : n \geq 2\}$ (see Fig. 14).



$\mathcal{W}_n$; $n \geq 2$

Fig. 14.

At this time, the authors conjectured that all 3-irreducible posets, all 3-interval irreducible posets of height one, and all graphs in $\mathcal{A}$ with clique covering number two were known. We now describe a combinatorial connection between comparability graphs and dimension theory which has allowed us to completely determine the collection $\mathcal{P}$ of all 3-irreducible posets and to subsequently confirm our conjectures. The connection is based on another test given by Dushnik and Miller [7] for posets with dimension at most two: $\dim(X) \leq 2$ iff $\bar{G}(X)$ is a comparability graph. It follows that if X is a 3-irreducible poset, then $\bar{G}(X)$ is not a comparability graph and that every proper induced subgraph of $\bar{G}(X)$ is a comparability graph.

Theorem 9. $\{\bar{G}(X): X \in \mathcal{P}\} \subset \mathcal{C}$.

We may then proceed to examine the graphs in $\mathcal{C}$ to see which ones have the property that their complements are comparability graphs. This process is simplified considerably since we have a reasonable conjecture for $\mathcal{P}$ and since an irreducible poset is determined (up to duality) by its comparability graph [36]. We conclude that $\mathcal{P}$ consists of seven infinite families: $\{C_n : n \geq 3\}$, $\{T_n : n \geq 1\}$, $\{D_n : n \geq 1\}$, $\{M_n : n \geq 1\}$, $\{N_n : n \geq 1\}$, $\{P_n : n \geq 1\}$, $\{W_n : n \geq 2\}$, and ten odd examples: $X_1, X_2, X_3, X_5, X_6, X_7, X_8, X_{11}, X_{12}$, and the chevron.

It is interesting to note that Kelly [15] has simultaneously and independently determined $\mathcal{P}$ by lattice theoretic methods. Since the completion by cuts of a poset of dimension t also has dimension t, Kelly determined $\mathcal{P}$ by finding the minimum collection of posets each of whose completion by cuts contains a lattice from $\mathcal{L}$.

Conversely it is very easy to determine $\mathcal{L}$ starting from $\mathcal{P}$ since the completions of the posets in $\mathcal{P}$ are easily describable. It should be noted that Kelly and Rival's determination of $\mathcal{L}$ does not depend on prior knowledge of $\mathcal{P}$.

We return now to the characterization problem for 3-interval irreducible posets. Although we have not been able to settle the general question, we have been able to use $\mathcal{P}$ to determine the subcollection of $\mathcal{P}_{\mathrm{I}}$ consisting of all 3-interval irreducible posets of height one. Our work is based on the following theorem.

Theorem 10. *Every t-interval irreducible poset of height one is a subposet of the split of a t-irreducible poset.*

Using this theorem we can announce the following result which confirms that $\mathcal{R}_{\mathrm{I}}$ consists precisely of the families listed in Table 3. Consequently, the determination of $\mathcal{A}_2$ is also complete.

Theorem 11. *Let X be a 3-irreducible poset and Y a 3-interval-irreducible subposet of* S(X).

(1) *If* $\mathcal{F}_Y = \mathcal{C}_i$ *for some* $i \geq 3$, *then* $X = C_i = S_3^{i-3}$.
(2) *If* $\mathcal{F}_Y = \mathcal{T}_1$, *then X is either* X_3 *or* T_1.
If $\mathcal{F}_Y = \mathcal{T}_i$, *for some* $i \geq 2$, *then* $X = T_i$.
(3) *If* $\mathcal{F}_Y = \mathcal{W}_1$, *then X is either the chevron* (Fig. 5) *or* X_6.
If $\mathcal{F}_Y = \mathcal{W}_i$ *for some* $i \geq 2$, *then* $X = W_i$.
(4) *If* $\mathcal{F}_Y = \mathcal{D}_1$, *then X is either* X_7, X_8, *or* D_1.
If $\mathcal{F}_Y = \mathcal{D}_i$ *for some* $i \geq 2$, *then* $X = D_i$.
(5) *If* $\mathcal{F}_Y = \mathcal{M}_i$ *for some* $i \geq 2$, *then X is either* M_i *or* P_i.
(6) *If* $\mathcal{F}_Y = \mathcal{N}_i$ *for some* $i \geq 1$, *then* $X = N_i$.
(7) *If* $\mathcal{F}_Y = \mathcal{G}_1$, *then* $X = X_1$.
(8) *If* $\mathcal{F}_Y = \mathcal{G}_2$, *then X is either* X_2, X_{11}, *or* X_{12}.
(9) *If* $\mathcal{F}_Y = \mathcal{G}_3$, *then* $X = X_5$.

Moore's original determination of $\mathcal{R}^{\Delta}$ was considerably more involved than the determination of $\mathcal{R}$. On the other hand, the determination of $\mathcal{R}_{\mathrm{I}}^{\Delta}$ is relatively simple. For a family $\mathcal{F}$, we denote $\mathcal{F} \cup \{A - B \colon A, B \in \mathcal{F}\}$ by $\Delta(\mathcal{F})$.

Theorem 12. $\mathcal{R}_{\mathrm{I}}^{\Delta} = \mathcal{R}^{\Delta}$.

Proof. Let $\mathcal{F} \in \mathcal{R}_{\mathrm{I}}^{\Delta}$. Since $\mathcal{F}$ is not Δ-interval-representable, it cannot be Δ-point-representable either. Choose a derived subfamily $\mathcal{G}$ of $\mathcal{F}$ with $\mathcal{G} \in \mathcal{R}^{\Delta}$. Now every proper derived subfamily of $\mathcal{G}$ is point-representable and therefore interval representable. It remains only to show that $\Delta(\mathcal{G})$ is not interval-representable for this implies $\mathcal{F} = \mathcal{G}$.

If $\mathcal{G} = \mathcal{C}_i$ for some $i \geq 3$, then $\Delta(\mathcal{G})$ also contains $\mathcal{C}_i$. If $\mathcal{G}$ is $\mathcal{F}_1, \mathcal{F}_3, \mathcal{F}_5$ then $\Delta(\mathcal{G})$ contains $\mathcal{C}_3$. If $\mathcal{G}$ is $\mathcal{F}_4$, then $\Delta(\mathcal{G})$ contains $\mathcal{C}_4$. If $\mathcal{G}$ is $\mathcal{F}_2$, then $\Delta(\mathcal{G})$ contains $\hat{\mathcal{G}}_3$. Finally, if $\mathcal{G}$ is $\mathcal{F}_6$, then $\Delta(\mathcal{G})$ contains $\mathcal{W}_1$.

It should be clear that Gallai's determination of $\mathcal{C}$ is a very powerful result since from it we can derive $\mathcal{P}, \mathcal{R}^{\Delta}, \mathcal{R}_{\mathrm{I}}, \mathcal{R}_{\mathrm{I}}^{\Delta}, \mathcal{A}_2$, and $\mathcal{L}$. Furthermore it is possible to determine $\mathscr{I}$ (and thus $\mathscr{R}$) from $\mathscr{C}$ via the following elementary reasoning. Let G be an interval graph; then $\bar{G}$ is the comparability graph of an interval order. Furthermore, if $G \in \mathscr{I} - \{C_4\}$, then $\bar{G}$ is not a comparability graph. For if $\bar{G}$ is the comparability graph of a poset X, since G does not contain C_4, we conclude, in view of Fishburn's characterization of interval orders, that X is an interval order; but this would imply that G is an interval graph.

Theorem 13. $\{\bar{G}: G \in \mathscr{I}, G \neq C_4\} \subset \mathscr{C}$.

We summarize our results by remarking that of the characterization problems discussed in this paper, only the circular arc graph and 3-interval irreducible poset problems remain unsolved, but we feel that the techniques and concepts developed here may well prove essential to their solution.

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