



The Graph of Critical Pairs of a Crown

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Received: 22 August 2017 / Accepted: 7 March 2019 / Published online: 2 April 2019
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Abstract

There is a natural way to associate with a poset P a hypergraph \mathcal{H} , called the hypergraph of critical pairs, so that the dimension of P is exactly equal to the chromatic number of \mathcal{H} . The edges of \mathcal{H} have variable sizes, but it is of interest to consider the graph G formed by the edges of \mathcal{H} that have size 2. The chromatic number of G is less than or equal to the dimension of P and the difference between the two values can be arbitrarily large. Nevertheless, there are important instances where the two parameters are the same, and we study one of these in this paper. Our focus is on a family $\{S_n^k : n \geq 3, k \geq 0\}$ of height two posets called crowns. We show that the chromatic number of the graph G_n^k of critical pairs of the crown S_n^k is the same as the dimension of S_n^k , which is known to be $\lceil 2(n+k)/(k+2) \rceil$. In fact, this theorem follows as an immediate corollary to the stronger result: The independence number of G_n^k is $(k+1)(k+2)/2$. We obtain this theorem as part of a comprehensive analysis of independent sets in G_n^k including the determination of the second largest size among the maximal independent sets, both the reversible and non-reversible types.

Keywords Dimension · Chromatic number · Critical pairs

1 Introduction

In this paper, we are concerned with a family of posets introduced in [14] called *crowns*. For a pair (n, k) of integers with $n \geq 3$ and $k \geq 0$, the *crown* S_n^k is the height 2 poset whose ground set¹ is $A \cup B$ where $A = \text{Min}(P) = \{a_1, a_2, \dots, a_{n+k}\}$ and $B = \text{Max}(P) = \{b_1, b_2, \dots, b_{n+k}\}$. Furthermore, a_i is incomparable with b_j when j belongs to the interval

¹In the original paper [14], A is the set of maximal elements and B is the set of minimal elements. Here these roles are reversed, a practice that is consistent with modern research papers in the area. Additionally, we use an entirely equivalent, but slightly modified labeling in order to considerably simplify the arguments to follow.

Much of the research done by Barrera-Cruz, Smith, and Taylor was completed while they were affiliated with the Georgia Institute of Technology.

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$\{i, i + 1, i + 2, \dots, i + k\}$ and $a_i < b_j$ in S_n^k when j is not in this interval. Of course, this definition must be interpreted cyclically, i.e., $n + k + 1 = 1$, $n + k + 2 = 2$, etc.

As will be detailed later, the critical pairs of S_n^k are precisely the incomparable pairs of the form (a_i, b_j) . The associated graph of critical pairs, G_n^k , has a vertex for each critical pair and an edge between (a_i, b_j) and (a_k, b_ℓ) if and only if $a_i < b_\ell$ and $a_k < b_j$ in S_n^k .

Originally, our goal was to prove the following theorem, a result that was conjectured by Garcia, Harris, Kubik, and Talbott [9] and launched the research collaboration represented by this manuscript.

Theorem 1.1 *Let (n, k) be a pair of integers with $n \geq 3$ and $k \geq 0$, and let G_n^k be the graph of critical pairs associated with the crown S_n^k . Then the chromatic number of G_n^k is equal to the dimension of S_n^k .*

As we explain in the next section, the preceding theorem follows as an immediate corollary to Theorem 1.2.

Theorem 1.2 *Let (n, k) be a pair of integers with $n \geq 3$ and $k \geq 0$, and let G_n^k be the graph of critical pairs associated with the crown S_n^k . Then the maximum size of an independent set in G_n^k is $(k + 1)(k + 2)/2$.*

We first managed to prove Theorem 1.1 *without* resolving Theorem 1.2, and we explain in Section 7 how this was achieved. However, in preparing early versions of this paper, we continued to build on our understanding of the properties of independent sets in G_n^k , and as a result, we were able to settle Theorem 1.2. In fact, this proof emerges as a minor detail extracted from a comprehensive body of results concerning independent sets in G_n^k .

First, independent sets in G_n^k will be classified as being one of two types: reversible or non-reversible. A reversible set is always independent, but in general, there are non-reversible sets that are also independent. A subset of a reversible set is reversible, and a subset of an independent set is independent, so there are natural notions of a maximal reversible set and a maximal independent set. Among the reversible sets, a special family is defined and members of this family are called canonical reversible sets. All canonical reversible sets have size $(k + 1)(k + 2)/2$ which is the maximum size for a reversible set, and the maximum size for an independent set. Here is the first of our three main theorems.

Theorem 1.3 *Let (n, k) be a pair of integers with $n \geq 3$ and $k \geq 0$. Then the following statements hold.*

- (1) *If S is a maximal reversible set in G_n^k , then S is a maximal independent set in G_n^k .*
- (2) *If $n > k$ and S is a maximal reversible set in G_n^k , then S is a canonical reversible set in G_n^k .*
- (3) *If S is a maximal reversible set in G_n^k and S is not a canonical reversible set, then $n \leq k$ and*

$$|S| \leq \frac{(k + 1)(k + 2)}{2} - \frac{n(n - 1)}{2} + 1.$$

Although it is possible to completely determine the family of all maximal reversible sets, our focus is on finding the second largest size such sets can attain. This detail is critical to subsequent arguments.

To set the stage for the study of independent, non-reversible sets, we prove the following elementary lemma in Section 4.

Lemma 1.4 *Let (n, k) be a pair of integers with $n \geq 3$ and $k \geq 0$. Then there is an independent non-reversible set in G_n^k if and only if $n \leq 2k$.*

As we begin to investigate independent, non-reversible sets, it will quickly become clear that there is a natural division into two ranges: $n \leq k$ and $k < n \leq 2k$. Independent, non-reversible sets in the range $k < n \leq 2k$ are relatively simple, and we completely determine the family of all maximal independent, non-reversible sets. Accordingly, we also establish the following upper bound and show that it is best possible. This result is the second of our three main theorems.

Theorem 1.5 *Let (n, k) be a pair of integers with $n \geq 3$ and $k \geq 0$. If $k < n \leq 2k$ and S is an independent, non-reversible set in G_n^k , then*

$$|S| \leq 2 + \frac{(2k + 2 - n)(2k + 1 - n)}{2}.$$

In particular, when $k < n \leq 2k$, there is a unique (up to isomorphism) independent, non-reversible set of size $2 + (2k + 2 - n)(2k + 1 - n)/2$.

The situation when $n \leq k$ is considerably more complex, and we are unable to determine the family of all maximal independent, non-reversible sets. However, we provide the following upper bound on their size in our third major theorem.

Theorem 1.6 *Let (n, k) be a pair of integers with $n \geq 3$ and $k \geq 0$. If $n \leq k$ and S is an independent, non-reversible set in G_n^k , then*

$$|S| \leq \frac{(k + 1)(k + 2)}{2} + 2 - n.$$

We show that the inequality in Theorem 1.6 is best possible. In fact, we have been able to completely characterize the family of all extremal examples. Due to the length of the details for the proof, we have elected to present a single representative example in this paper.

It is worth noting that the inequality from Theorem 1.6 holds for all pairs (n, k) , since when $k < n \leq 2k$ we have:

$$\begin{aligned} 2 + \frac{(2k + 2 - n)(2k + 1 - n)}{2} &= \frac{(k + 1)(k + 2)}{2} + 2 - n - \frac{(3k - n)(n - k - 1)}{2} \\ &\leq \frac{(k + 1)(k + 2)}{2} + 2 - n. \end{aligned}$$

However, this inequality is strict when $n > k + 1$.

Once these theorems have been proved, it is simply a remark that for all pairs (n, k) with $n \geq 3$ and $k \geq 0$, the maximum size of an independent set in G_n^k is $(k + 1)(k + 2)/2$, an observation which is enough to show that $\dim(S_n^k) = \chi(G_n^k)$. However, we have proved much more since we have identified the canonical reversible sets as the maximum size independent sets, and we have found the maximum size for all other maximal independent sets, for both the reversible and non-reversible cases.

The remainder of this paper is organized as follows. In the next section, we provide notation and terminology together with a concise summary of background material to motivate this line of research. In Section 3, we study reversible sets and prove Theorem 1.3. Independent, non-reversible sets are more complex and Section 4 is an introductory section in which essential proof techniques are developed and the proof of Lemma 1.4 is given. Sections 5 and 6 are devoted to the proofs of the inequalities in Theorems 1.5 and 1.6, respectively.

In Section 7, we provide details on how Theorem 1.1 can be proved directly without Theorem 1.2. These techniques are of independent interest, even if the conclusion is attainable by the results presented earlier in the paper.

We close in Section 7 with some comments on challenging open problems that remain.

2 Notation, Terminology, and Background Material

We assume that the reader is familiar with basic notation and terminology for partially ordered sets (here we use the short term poset), including: chains and antichains, minimal and maximal elements, linear extensions, and comparability graphs. While we are also assuming some level of familiarity with the concept of dimension for posets, there are a number of recent papers, [5, 10, 19], each of which contains a more complete discussion of the necessary background material. A comprehensive treatment is given in [15], and a survey of combinatorial aspects of posets is given in [17], so we include here only the essential definitions.

A non-empty family $\mathcal{F} = \{L_1, L_2, \dots, L_d\}$ of linear extensions of a poset P is called a *realizer* of P when $x \leq y$ in P if and only if $x \leq y$ in L_i for each $i \in \{1, 2, \dots, d\}$. As defined by Dushnik and Miller [3], the *dimension* of P , denoted $\dim(P)$, is the least positive integer d for which P has a realizer of size d .

When P is a poset, we let $\text{Inc}(P)$ be the set of all pairs $(x, y) \in P \times P$ with x incomparable to y in P . Clearly, a non-empty family \mathcal{F} of linear extensions of P is a realizer if and only if for every $(x, y) \in \text{Inc}(P)$, $x > y$ in L for some $L \in \mathcal{F}$ due to the symmetry of pairs in $\text{Inc}(P)$.

A subset $R \subset \text{Inc}(P)$ is said to be *reversible* when there is a linear extension L of P with $x > y$ in L for all $(x, y) \in R$. Accordingly, when $\text{Inc}(P) \neq \emptyset$, $\dim(P)$ is the least d for which there is a covering $\text{Inc}(P) = R_1 \cup R_2 \cup \dots \cup R_d$ with R_i reversible for each $i \in \{1, 2, \dots, d\}$.

An indexed set $C = \{(x_\alpha, y_\alpha) : 1 \leq \alpha \leq m\} \subseteq \text{Inc}(P)$ of incomparable pairs in P is called an *alternating cycle* of size m when² $x_\alpha \leq y_{\alpha-1}$ in P , for all $\alpha \in \{1, 2, \dots, m\}$. (Subscripts are interpreted cyclically so that $x_1 \leq y_m$ in P .) An alternating cycle is *strict* when $x_\alpha \leq y_\beta$ if and only if $\beta = \alpha - 1$. In a strict alternating cycle, the set $\{x_\alpha : 1 \leq \alpha \leq m\}$ is an m -element antichain in P as is $\{y_\alpha : 1 \leq \alpha \leq m\}$.

Although the proof of the following lemma, first presented in [18], is elementary, the basic ideas behind this result have proven over time to be very important.

Lemma 2.1 *Let P be a poset and let $S \subseteq \text{Inc}(P)$. Then the following statements are equivalent.*

- (1) S is not reversible.
- (2) S contains an alternating cycle.
- (3) S contains a strict alternating cycle.

We note the following property of alternating cycles: If $C = \{(x_\alpha, y_\alpha) : 1 \leq \alpha \leq m\}$ is an alternating cycle, but is not strict, then there is a proper subset of the pairs in C which (after a relabeling) forms a strict alternating cycle.

²Most authors require that $x_\alpha \leq y_{\alpha+1}$ in P in defining an alternating cycle. Our equivalent formulation is another choice that simplifies arguments to follow.

A pair $(x, y) \in \text{Inc}(P)$ is called a *critical pair* if (1) $z < y$ in P whenever $z < x$ in P and (2) $w > x$ whenever $w > y$ in P . We let $\text{Crit}(P)$ denote the set of all critical pairs. Interest in critical pairs is rooted in the well known property that a non-empty family \mathcal{F} of linear extensions is a realizer if and only if for every pair $(x, y) \in \text{Crit}(P)$, there is some $L \in \mathcal{F}$ with $x > y$ in L . When $\text{Inc}(P) \neq \emptyset$, $\dim(P)$ is the least positive integer d for which there is a covering $\text{Crit}(P) = R_1 \cup R_2 \cup \dots \cup R_d$ with R_i reversible for each $i \in \{1, 2, \dots, d\}$.

Although the origins can be traced back to earlier papers, it seems that the first concrete formulation of the following concept appears in [11]: Given a poset P for which $\text{Inc}(P) \neq \emptyset$, we can associate with P a hypergraph \mathcal{H} and a graph G of critical pairs defined as follows. Both \mathcal{H} and G have the set $\text{Crit}(P)$ of critical pairs as their vertex set. In \mathcal{H} , a set E of critical pairs is an edge when E is not reversible, but every proper subset of E is reversible. The edge set of the graph G is just the set of all edges in \mathcal{H} which have size 2. In view of the remarks made immediately above, $\dim(P) = \chi(\mathcal{H})$ when $\text{Inc}(P) \neq \emptyset$, where $\chi(H)$ is the minimum k such that there is a k -coloring of the vertices of H with no monochromatic edge. On the other hand, we only have the inequality $\dim(P) \geq \chi(G)$.

In [5], an infinite sequence $\{P_n : n \geq 1\}$ of posets is constructed such that the dimension of P_n grows exponentially with n while the chromatic number of the graph of critical pairs grows linearly with n . Accordingly, the inequality $\dim(P) \geq \chi(G)$ can be far from tight. Nevertheless, it is of interest to investigate conditions which cause $\dim(P)$ to be equal to $\chi(G)$.

The research we report concerns a well studied class of posets called crowns. The first use of the term “crown” in reference to a class of posets is in [1], where it is applied only to posets in the subfamily $\{S_3^k : k \geq 0\}$. These posets are all 3-irreducible, i.e., they have dimension 3, but the removal of any point lowers the dimension to 2. This special case plays an important role in the on-line notion of dimension (see [6] and [12]). Also, the family of crowns includes the *standard examples*. These are the posets in the family $\{S_n^0 : n \geq 3\}$. For each $n \geq 3$, the crown S_n^0 is n -irreducible, and in the literature, the notation for the standard example S_n^0 is usually abbreviated as S_n .

The fact that $\dim(S_3^k) = 3$ for all $k \geq 0$ and $\dim(S_n^0) = n$ for all $n \geq 3$ suggests that there may be a function $f(d)$ such that if the maximum degree in the comparability graph of P is d , then the dimension of P is at most $f(d)$. In fact, these examples suggest that it might even be true that $f(d) = d + 1$. But for many years, it was not even known whether the function $f(d)$ was well defined. However, Füredi and Kahn [8] proved that $f(d) = O(d \log^2 d)$ and Erdős, Kierstead and Trotter [4] proved that $f(d) = \Omega(d \log d)$. Very recently, Scott and Wood [13] significantly improved the upper bound using the notion of boxicity, showing $f(d) = O(d \log^{1+o(1)} d)$. So the original interest in the family of crowns was to see if these posets shed further light on the problem of dimension versus maximum degree in the comparability graph.

Most authors extend the notion of the standard example S_n to the value $n = 2$, i.e., S_2 is the poset of height 2 with minimal elements $\{a_1, a_2\}$, maximal elements $\{b_1, b_2\}$, and $a_i < b_j$ in S_2 if and only if $i \neq j$. The standard example S_2 is just $\mathbf{2} + \mathbf{2}$, the disjoint sum of two 2-element chains, with all points of one chain incomparable with all points in the other. Posets which exclude $S_2 = \mathbf{2} + \mathbf{2}$ are called *interval orders*, since if P is such a poset, there is family $\mathcal{F} = \{[c_x, d_x] : x \in P\}$ of non-degenerate closed intervals of the real line so that $x < y$ in P if and only if $d_x < c_y$ in the reals. The class of interval orders has been studied extensively in the literature. See [7] and [16] for results and references. Note that S_2^0 has dimension 2, but it is not 2-irreducible.

Based on the examples in the families $\{S_3^k : k \geq 0\}$ and $\{S_n^0 : n \geq 3\}$, it was originally thought that it might be possible that $\dim(S_n^k) = n$ for all pairs (n, k) with $n \geq 3$ and $k \geq 0$. Some small cases not belonging to these families were worked out by hand, leading first to the conclusion that $\dim(S_4^1) = 4$ which fit the suspected pattern. But subsequently, it was shown that $\dim(S_4^2) = 3$, so the pattern does not hold in general. These observations then motivated an attack on finding the general form for the dimension of the crown S_n^k , which led eventually to the following formula, given in [14].

Theorem 2.2 *Let (n, k) be a pair of integers with $n \geq 3$ and $k \geq 0$. Then $\dim(S_n^k) = \lceil 2(n+k)/(k+2) \rceil$.*

Clearly, the critical pairs in the crown S_n^k are just the pairs $(a, b) \in A \times B$ with a incomparable to b in S_n^k . We denote the set of such pairs by $\text{Inc}(A, B)$. There are $(n+k)(k+1)$ critical pairs in $\text{Inc}(A, B)$, so the inequality $\dim(S_n^k) \geq 2(n+k)/(k+2)$ is an immediate consequence of the following lemma, which appears (with different notation) on page 92 in [14].

Lemma 2.3 *The maximum number of critical pairs that can be reversed by a linear extension of the crown S_n^k is $(k+1)(k+2)/2$.*

In retrospect, it is fair to say that the argument presented in [14] is incomplete. However, the lemma also appears on pages 33 and 34 in [15], and the proof given there is complete and correct. As it only takes a few lines and serves to set the stage for a more subtle result to follow, we include an updated proof of Lemma 2.3 in Section 3.

Recall that when G is a graph, a subset S of the vertex set of G is called an *independent set* when no two vertices in S are adjacent in G . The *independence number* of G , denoted $\alpha(G)$, is then defined as the maximum cardinality of an independent set in G , and the *chromatic number* of G , denoted $\chi(G)$, is the least positive integer t for which the vertex set of G can be partitioned into t independent sets. If G has m vertices, then $\chi(G) \geq m/\alpha(G)$. This implies $\dim(S_n^k) \leq \chi(G_n^k)$, since the graph G_n^k of critical pairs of the crown S_n^k has $(n+k)(k+1)$ vertices and $\dim(S_n^k) = 2(n+k)/(k+2)$. As the opposite inequality holds for all graphs, Theorem 1.1 is now seen to be a corollary to Theorem 1.2.

A reversible set in G_n^k is an independent set, but in general there are independent sets which are not reversible. These are sets of critical pairs that contain one or more alternating cycles, but none of size 2. Lemma 2.3 asserts that the maximum size of a reversible set in G_n^k is $(k+1)(k+2)/2$, leaving open the possibility that there are independent, non-reversible sets which have size larger than $(k+1)(k+2)/2$. However, as is clear from our series of main theorems, we show that this is not the case.

Convention For the remainder of the paper, the symbols n and k will only be used in reference to the crown S_n^k and the associated graph G_n^k of critical pairs. Accordingly, we always assume $n \geq 3$ and $k \geq 0$.

2.1 Canonical Reversible Sets

We say that a subset X of minimal elements A is *contiguous* when the elements of X form a block of consecutive elements of A with indices interpreted cyclically. For example, the set $X = \{a_8, a_1, a_9, a_2\}$ is contiguous in S_4^5 . Vacuously, both \emptyset and A are contiguous. Contiguous subsets of the maximal elements B are defined analogously.

A sequence $\sigma = (x_1, x_2, \dots, x_r)$ of distinct elements of A will be called an *h-contiguous sequence* when $X_i = \{x_1, x_2, \dots, x_i\}$ is contiguous for all $i \in \{1, 2, \dots, r\}$. For example, in the crown S_4^5 , the sequence $\sigma = (a_8, a_9, a_7, a_1, a_6, a_2)$ is *h-contiguous*. The letter *h* in this notation stands for “hereditarily.”

When $\sigma = (a_1, a_2, \dots, a_r)$ is an *h-contiguous sequence*, we let $T(\sigma)$ consist of all pairs $(a_i, b) \in \text{Inc}(A, B)$ and, for each $1 \leq i < r$, we include (a_{i+1}, b) in $T(\sigma)$ provided $(a_{i+1}, b) \in \text{Inc}(A, B)$ and $(a_i, b) \in T(\sigma)$.

The inequality in Lemma 2.3 is easily seen to be tight, as evidenced by the construction in the following lemma, which is implicit in the results of [14].

Lemma 2.4 *Let (n, k) be a pair of integers with $n \geq 3$ and $k \geq 0$. If $\sigma = (x_1, x_2, \dots, x_{k+1})$ is an *h-contiguous sequence*, then $T(\sigma)$ is reversible, $T(\sigma)$ is a maximal independent set in G_n^k , and $|T(\sigma)| = (k + 1)(k + 2)/2$.*

In the discussions to follow, we say that an independent set T in G_n^k is a *canonical reversible set* when there is an *h-contiguous sequence* σ for which $T = T(\sigma)$.

Example 2.5 *To illustrate the preceding lemma and the connection between h-contiguous sequences and canonical reversible sets, consider the crown S_4^5 . Then $\sigma = (a_8, a_9, a_7, a_1, a_6, a_2)$ is an *h-contiguous sequence* of length $6 = k + 1$. The canonical reversible set $T = T(\sigma)$ associated with σ is:*

$$T = \{ (a_8, b_8), (a_8, b_9), (a_8, b_1), (a_8, b_2), (a_8, b_3), (a_8, b_4), \\ (a_9, b_9), (a_9, b_1), (a_9, b_2), (a_9, b_3), (a_9, b_4), \\ (a_7, b_9), (a_7, b_1), (a_7, b_2), (a_7, b_3), \\ (a_1, b_1), (a_1, b_2), (a_1, b_3), \\ (a_6, b_1), (a_6, b_2), \\ (a_2, b_2) \}.$$

The linear extension of S_4^5 represented by the following sequence, where the order is increasing left to right, reverses the $6 \cdot 7/2 = 21$ pairs in T :

$$(a_3, a_4, a_5, b_2, a_2, b_1, a_6, b_3, a_1, b_9, a_7, b_4, a_9, b_8, a_8, b_5, b_6, b_7).$$

Previously, we observed that the inequality $\dim(S_n^k) \geq \lceil 2(n + k)/(k + 2) \rceil$ follows from the fact that no linear extension of S_n^k can reverse more than $(k + 1)(k + 2)/2$ critical pairs. In [14], the reverse inequality is proved by showing that the set of all critical pairs of S_n^k can be covered by $\lceil 2(n + k)/(k + 2) \rceil$ canonical reversible sets. While a simple construction shows $\dim(S_n^k) \leq 2\lceil (n + k)/(k + 2) \rceil$, the improvement to $\dim(S_n^k) \leq \lceil 2(n + k)/(k + 2) \rceil$ takes a bit of work.

2.2 Special Notation and Terminology for Crowns

As sets A and B are the minimal elements and maximal elements, respectively, of S_n^k , the letter a will always refer to a minimal element, while b is reserved for maximal elements. In order to avoid confusion over subscripts, we also use the letters x and z , sometimes with subscripts or primes, to denote elements of A , while the letters y and w always represent elements of B . On the other hand, the letter v is used to represent an element of S_n^k which may come from either A or B .

Following the conventions of the presentation in [15], we consider a circle in the Euclidean plane with $n + k$ evenly spaced points on the circle labeled in clockwise order $(u_1, u_2, \dots, u_{n+k})$. We then imagine the elements of the crown S_n^k placed on this circle with a_i and b_i both positioned on the point u_i .

If $u_i, u_j,$ and u_k are distinct points on the circle, then we write $u_i < u_j < u_k$ to signify that if we traverse the circle in a clockwise direction, starting from u_i and stopping the first time we encounter u_k , then somewhere in between, we saw u_j .³ This notation can be used with inequalities which are not strict, so the statement $u_i < u_j \leq u_k$ implies that $u_i \neq u_j$ and $u_i \neq u_k$ while u_j and u_k could be the same.

We extend this definition to include all elements of S_n^k . In particular, for distinct elements $v, v',$ and v'' of S_n^k , we write $v < v' < v''$ if $u < u' < u''$ where v is positioned at point u on the circle, v' at u' and v'' at u'' . Recall the convention that $a, x \in A$ and $y \in B$. So a statement like $x < y \leq a$ includes the possibility that $y = b_i$ and $a = a_i$ for some i .

When v and v' are elements of S_n^k , v located at position u_i and v' at u_j , then we say the *size* of (v, v') is $j - i + 1$, modulo $n + k$. Note that this is simply the number of points visited on the circle (including the beginning and ending points) when we travel from v to v' moving in a clockwise manner. For example, in S_4^5 , the size of (a_7, b_1) is 4 and the size of (b_6, a_8) is 3.

When $v_1, v_2 \in A \cup B$, it is natural to say that (v_1, v_2) starts at v_1 and ends at v_2 . For $v_3, v_4 \in A \cup B$, we say (v_1, v_2) starts in (v_3, v_4) if $v_3 \leq v_1 \leq v_4$ and ends in (v_3, v_4) when $v_3 \leq v_2 \leq v_4$.

It is also natural to consider $\text{Inc}(A, B)$ as an inclusion order, where we say that (a, b) is contained in (x, y) if $x \leq a \leq b \leq y$. Further, (a, b) overlaps (x, y) , if there is some point u on the circle so that $a \leq u \leq b$ and $x \leq u \leq y$. If (a, b) and (x, y) do not overlap, we say they are disjoint. Note that (a, b) and (x, y) are adjacent in G_n^k if and only if they are disjoint and both (b, x) and (y, a) have size at most n .

Throughout the paper, we use the now standard notation $[m]$ to represent the set $\{1, 2, \dots, m\}$.

2.3 Natural Symmetries of the Crown S_n^k

The crown S_n^k and the graph G_n^k have two kinds of natural symmetries. One of these captures the notion of rotation. For each j , the map τ_j defined by setting $\tau_j(a_i) = a_{i+j}$ and $\tau_j(b_i) = b_{i+j}$ is both an automorphism of the crown S_n^k and an automorphism of the graph G_n^k .

However, there is another natural symmetry that essentially results from interchanging clockwise with counter-clockwise. In particular, the map $\phi : A \cup B \rightarrow A \cup B$ defined by $\phi(a_i) = a_{-i}$ and $\phi(b_j) = b_{k-j}$ is both an automorphism of the crown S_n^k and an automorphism of the graph G_n^k . To see this, we observe that $(a_i, b_j) \in \text{Inc}(A, B)$ if and only if $u_i \leq u_j \leq u_{i+k}$. Since $u_0 = u_{n+k}$, we have the following string of equivalent inequalities:

$$\begin{aligned} u_i &\leq u_j \leq u_{i+k}, \\ u_0 &\leq u_{j-i} \leq u_k, \\ u_0 &\leq u_{k-(j-i)} \leq u_k, \\ u_{-i} &\leq u_{k-j} \leq u_{-i+k}. \end{aligned}$$

³When we write $u_{j_1} < u_{j_2} < u_{j_3} < \dots < u_{j_\ell}$, we intend that when traversing the circle in a clockwise direction, starting from u_{j_1} and stopping the first time we encounter u_{j_ℓ} , we visit $u_{j_2}, u_{j_3}, \dots, u_{j_{\ell-1}}$ in that order.

Moreover, if the size of the pair (a, b) is s , then the size of $(\phi(a), \phi(b))$ is $k + 2 - s$. As ϕ induces a bijection on $A \times B$, this implies $(a, b) \in \text{Inc}(A, B)$ if and only if $(\phi(a), \phi(b)) \in \text{Inc}(A, B)$. Also note that if $(a, b), (x, y) \in \text{Inc}(A, B)$, then (a, b) is contained in (x, y) if and only if $(\phi(x), \phi(y))$ is contained in $(\phi(a), \phi(b))$.

When $S \subseteq \text{Inc}(A, B)$, we let $\phi(S)$ denote the set $\{(\phi(x), \phi(y)) : (x, y) \in S\}$. Clearly, (1) $|S| = |\phi(S)|$, (2) S is independent if and only if $\phi(S)$ is independent and (3) S is reversible if and only if $\phi(S)$ is reversible.

2.4 Types of Independent Sets

Recall that a reversible set is always an independent set. However, in general the inclusion is strict as there may be independent sets that are not reversible. For the rest of the paper, we follow the conventions that (1) the letter T is used for independent sets known to be canonical reversible sets; (2) the letter R is used for independent sets known to be reversible (such sets may or may not be canonical); and (3) the letter S is used for independent sets when they are either non-reversible or the issue of whether they are reversible has not been settled. To avoid possible confusion with crowns, we denote independent sets as S, S' or S'' , but we never use subscripts.

For a pair of integers (n, k) with $n \geq 3$ and $k \geq 0$, we use the following notation concerning families of subsets of the pairs in G_n^k .

- (1) $\mathbb{I}(n, k)$ is the family all independent sets.
- (2) $\text{MII}(n, k)$ is the family of all maximal independent sets.
- (3) $\text{MSI}(n, k)$ is the family of all maximum size independent sets.
- (4) $\mathbb{R}(n, k)$ is the family of all reversible sets.
- (5) $\text{MR}(n, k)$ is the family of all maximal reversible sets.
- (6) $\text{MSR}(n, k)$ is the family of all maximum size reversible sets.
- (7) $\text{INR}(n, k)$ is the family of all independent, non-reversible sets.
- (8) $\text{MINR}(n, k)$ is the family of all maximal sets in $\text{INR}(n, k)$.
- (9) $\text{MSINR}(n, k)$ is the family of all maximum size sets in $\text{INR}(n, k)$.

When the pair (n, k) has been fixed, we abbreviate the names of the above sets by leaving off the (n, k) . For example, $\mathbb{I}(n, k)$ is abbreviated \mathbb{I} .

Fix a pair (n, k) . When $S \in \mathbb{I}$, let $A(S)$ consist of all elements $a \in A$ for which there is some $b \in B$ with $(a, b) \in S$. The set $B(S)$ is defined analogously. For each $a \in A$, let $B(a, S)$ denote the set of all $b \in B$ for which $(a, b) \in S$. For $b \in B$, the set $A(b, S)$ is defined analogously. When $S = \text{Inc}(A, B)$, set $I(a) := B(a, S)$ and $I(b) := A(b, S)$. Of course, $I(a_i) = \{b_i, b_{i+1}, \dots, b_{i+k}\}$ and $I(b_j) = \{a_j, a_{j-1}, \dots, a_{j-k}\}$. Further, for any $S \in \mathbb{I}$, $B(a, S) \subseteq I(a)$, $A(b, S) \subseteq I(b)$. If $a \notin A(S)$, then $B(a, S) = \emptyset$. Likewise $A(b, S) = \emptyset$ when $b \notin B(S)$.

3 Reversible Sets

This section is devoted to the study of reversible sets and includes the proof of Theorem 1.3, the first of our three main theorems. As there are three statements in this theorem, we will remind readers of the wording of the individual statements at the appropriate moment in the argument.

Fix a pair (n, k) . Let $R \in \text{MIR}$ and let L be a linear extension of S_n^k which reverses all pairs in R . Scan L from bottom to top and note that L is a linear order on $A \cup B$ consisting

of blocks of elements of A interspersed with blocks of elements of B . The bottom block is $A - A(R)$ while the top block is $B - B(R)$. Accordingly, L has the following *block-structure* form, where the order of the blocks is increasing left to right:

$$(A_{s+1}, B_s, A_s, B_{s-1}, A_{s-1}, B_{s-2}, \dots, A_3, B_2, A_2, B_1, A_1, B_0).$$

In particular, $A_{s+1} = A - A(R)$ and $B_0 = B - B(R)$.

It is easy to see that any linear extension L that reverses R has this form; furthermore, the only allowable variation is the linear order imposed on elements within a block as R is maximal. The set R determines the block structure triple $(s, \mathcal{F}, \mathcal{G})$: the integer s and the two set partitions $\mathcal{F} = \{A_1, A_2, \dots, A_{s+1}\}$ and $\mathcal{G} = \{B_0, B_1, \dots, B_s\}$ of A and B , respectively.

We note that since L is a linear extension of P , it satisfies the following condition.

Admissibility Condition. If $x \in A_i, y \in B_j$, and $x < y$ in S_n^k , then $0 \leq j < i \leq s + 1$.

Since R is maximal, L also satisfies the following condition.

Maximality Condition. For each $i \in \{0, 1, \dots, s\}$ if $x \in A_{i+1}$ and $y \in B_i$, then $x < y$ in S_n^k .

To see that the Maximality Condition is satisfied, suppose there is some i and a pair $(x, y) \in \text{Inc}(A, B)$ with $x \in A_{i+1}$ and $y \in B_i$ where it is possible that $A_{i+1} = \{x\}$ or $B_i = \{y\}$. Then a linear extension with the following block form reverses all pairs in $R \cup \{(x, y)\}$, contradicting the assumption that $R \in \text{MIR}$:

$$(A_{s+1}, B_s, A_s, \dots, B_{i+1}, A_{i+1} - \{x\}, \{y\}, \{x\}, B_i - \{y\}, A_i, \dots, B_1, A_1, B_0).$$

We pause here to prove the first statement of Theorem 1.3: If $R \in \text{MIR}$, then $R \in \text{MII}$. Let $(x, y) \in \text{Inc}(A, B) - R$ be arbitrary. We argue that there exists $(z, w) \in R$ such that (z, w) is adjacent to (x, y) in G_n^k . Since R satisfies the maximality condition, there are integers i, j with $0 \leq i \leq j - 2 \leq s - 1$ so that $x \in A_j$ and $y \in B_i$. Let w be any element of B_{j-1} and let z be any element of A_{i+1} . It follows that the incomparable pair (z, w) is reversed and thus is in R . By the maximality condition, $x < w$ and $z < y$. Therefore, (x, y) and (z, w) are adjacent in G_n^k . This completes the proof.

In some sense, we now have characterized the sets in MIR , as they are exactly the sets R associated with a block structure triple $(s, \mathcal{F}, \mathcal{G})$ satisfying the admissibility and maximality conditions. However, in the work to follow, we need to know the largest two sizes these sets can have.

We note that when R is maximal reversible, if $a, a' \in A(R)$, then one of $B(a, R)$ and $B(a', R)$ is a subset of the other. Also, it may happen that $B(a, R) = B(a', R)$. Analogous remarks apply when $b, b' \in B$.

When $R \in \text{MIR}$, it is easy to see that $|A_{s+1}| = |B_0| = n - 1, |A_1| = |B_s| = 1$, and $|A(R)| = |B(R)| = k + 1$. Furthermore, R is a canonical reversible set if and only if $s = k + 2$. In this case, $|A_i| = |B_{i+1}| = 1$ for all $i \in [k + 1]$.

Again, let $R \in \text{MIR}$. A labeling $\{x_1, x_2, \dots, x_{k+1}\}$ of $A(R)$ is called a *consistent* labeling if $\alpha < \beta$ whenever $B(x_\beta, R)$ is a subset of $B(x_\alpha, R)$. One such labeling can be obtained from any ordering of $A(R)$ with block structure $(A_1, A_2, \dots, A_s, A_{s+1})$ ordered left to right. Here is an updated version of the proposition at the heart of the proof of Lemma 2.3, as given in [15].

Proposition 3.1 *Let $R \in \text{MIR}$ and let $\{x_1, x_2, \dots, x_{k+1}\}$ be a consistent labeling of $A(R)$. Then for each $i \in [k + 1], |B(x_i, R)| \leq k + 2 - i$.*

Proof The inequality holds (and is tight) when $i = 1$, so we may assume that $1 < i \leq k + 1$.

Let b_t be any element of $B(x_i, R)$. Then $\{x_1, x_2, \dots, x_i\} \subseteq A(b_t, R)$ by definition of a consistent labeling. Therefore, we can relabel the elements of $\{x_1, x_2, \dots, x_i\}$ as $\{z_1, z_2, \dots, z_i\}$ so that $a_{t-k} \leq z_1 < z_2 < \dots < z_i \leq b_t$. Again by the definition of a consistent labeling, $B(x_i, R) \subseteq B(z_1, R) \subseteq I(z_1)$. Now we find $i - 1$ elements in $I(z_1)$ which are not in $B(x_i, R)$ to bound the size of $B(x_i, R)$.

For each $j \in \{2, 3, \dots, i\}$, if $z_j = a_{\beta_j}$, then $b_{\beta_{j-1}} \in I(z_1)$ because $a_{t-k} \leq z_1 < a_{\beta_j} \leq b_t$. However, $z_j = a_{\beta_j} < b_{\beta_{j-1}}$ in S_n^k , which implies that $b_{\beta_{j-1}} \notin B(x_j, R)$. Because $B(z_i, R) \subseteq B(z_j, R)$, we have $b_{\beta_{j-1}} \notin B(z_i, R)$. Since $b_{\beta_{j-1}} \in I(z_1) - B(z_i, R)$ for each $j \in \{2, 3, \dots, i\}$ and $B(x_i, R) \subseteq I(z_1)$, we have $|B(x_i, R)| \leq (k + 1) - (i - 1) = k + 2 - i$. \square

Lemma 2.3 follows immediately from the preceding proposition since we have

$$|R| = \sum_{i=1}^{k+1} |B(x_i, R)| \leq \sum_{i=1}^{k+1} k + 2 - i = \frac{(k + 1)(k + 2)}{2}.$$

Although the inequality in Proposition 3.1 is tight for canonical reversible sets, if $R \in \mathbb{MR}$ and R is not canonical, it is not tight. For such sets, we have the following stronger result.

Lemma 3.2 *Let R be any set in \mathbb{MR} which is not a canonical reversible set, and let $\{x_1, x_2, \dots, x_{k+1}\}$ be a consistent labeling of $A(R)$. Choose i to be the least positive integer for which $|B(x_i, R)| \neq k + 2 - i$. Then the following hold:*

- (1) $i \geq 2$ and $i \leq k - n + 2$;
- (2) $|B(x_i, R)| = k + 4 - i - n$; and
- (3) For every $j \in \{i + 1, i + 2, \dots, k + 1\}$, $|B(x_j, R)| \leq k + 3 - i - n$.

Proof We have already noted $|B(x_1, R)| = k + 1$, so $i \geq 2$. The assumption that $|B(x_j, R)| = k + 2 - j$ for each $j \in [i - 1]$ implies that the sequence $(x_1, x_2, \dots, x_{i-1})$ is h -contiguous. The assumption $|B(x_i, R)| \neq k + 2 - i$ implies $\{x_1, x_2, \dots, x_i\}$ is not contiguous. After relabeling, we may assume $\{x_1, x_2, \dots, x_{i-1}\} = \{a_1, a_2, \dots, a_{i-1}\}$, and further $x_i \notin \{a_{n+k}, a_i\}$. By the definition of a consistent labeling, $B(x_{i-1}, R) \subseteq B(a_j, R)$ for each $j \in [i - 1]$. Therefore $B(x_{i-1}, R) \subseteq B(a_1, R) \cap B(a_{i-1}, R) \subseteq \{b_1, \dots, b_{k+1}\} \cap \{b_{i-1}, \dots, b_{i-1+k}\} = \{b_{i-1}, \dots, b_{k+1}\}$. Since $|B(x_{i-1}, R)| = k - i + 3$, we conclude that $B(x_{i-1}, R) = \{b_{i-1}, b_i, \dots, b_{k+1}\}$.

Our aim is to show $\{b_{i-1}, b_{k+1}\} \subseteq I(x_i)$. By the consistent labeling, $B(x_i, R) \subseteq B(x_{i-1}, R) = \{b_{i-1}, b_i, \dots, b_{k+1}\}$ with $B(x_i, R) \neq \emptyset$ as $x_i \in A(R)$.

For contradiction, first suppose $x_i < b_{i-1}$ and $x_i < b_{k+1}$. Therefore $I(x_i) \subseteq \{b_i, \dots, b_k\}$. Thus $|I(x_i)| < k + 1$, a contradiction. So at most one of $x_i < b_{k+1}$ and $x_i < b_{i-1}$ is possible.

Toward a contradiction, suppose $x_i < b_{i-1}$ while $b_{k+1} \in I(x_i)$. Since $x_i \notin \{a_1, a_2, \dots, a_i\}$ and $b_{k+1} \in I(x_i)$, then $x_i \in \{a_{i+1}, \dots, a_{k+1}\}$. In particular, $x_i < b_i$. In the block structure, $x_i \in A_i$ and $\{b_{i-1}, b_i\} \subseteq B_{i-1}$. Then $(a_i, b_i) \notin R$. Now if $a_i < b_j$ then either $a_{i-1} < b_j$ or $b_j = b_{i-1}$. So, even if a_i is not in block A_i , we can replace the blocks (A_i, B_{i-1}) with $(A_i - \{a_i\}, \{b_i\}, \{a_i\}, B_{i-1} - \{b_i\})$, which implies $R \cup (a_i, b_i)$ is reversible, contradicting the maximality of R . It can also be shown that $x_i < b_{k+1}$ and $b_{i-1} \in I(x_i)$ cannot occur by a similar argument. In that case, we contradict the maximality of R by showing that $R \cup (a_{n+k}, b_k)$ is reversible.

We now know that $\{b_{i-1}, b_{k+1}\} \subseteq I(x_i)$. Since $B(x_i, R) \subsetneq \{b_{i-1}, b_i, \dots, b_{k+1}\}$, it follows that the set of $n - 1$ elements, which x_i is comparable to, is a subset of $\{b_i, \dots, b_k\}$. Therefore $n - 1 \leq k + 1 - i$, which is equivalent to the first statement of the lemma. In particular, $|B(x_i, R)| = (k + 3 - i) - (n - 1)$, the second statement of the lemma.

For the third statement, suppose $x_i = a_m$, fix $j \in \{i + 1, i + 2, \dots, k + 1\}$, and suppose $x_j = a_p$. Note that $B(x_j, R) \subseteq B(x_i, R)$. In each of three different cases based on the value of p , we show that $B(x_i, R) - B(x_j, R) \neq \emptyset$. If $i \leq p \leq m - n$ or $m + 1 \leq p \leq k + 1$, then $b_{p-1} \in B(x_i, R) - B(x_j, R)$. If $m - n + 1 \leq p \leq m - 1$ then $b_{m-n} \in B(x_i, R) - B(x_j, R)$. If $k + 2 \leq p \leq n + k$, then $b_{k+1} \in B(x_i, R) - B(x_j, R)$. So in any case, $B(x_i, R) - B(x_j, R) \neq \emptyset$ and therefore $|B(x_j, R)| \leq k + 3 - i - n$. \square

The first statement of Lemma 3.2 yields the proof of the second statement in Theorem 1.3, i.e., if $n > k$ and $S \in \mathbb{MR}$, then S is a canonical reversible set.

Lemma 3.2 also supplies the proof of the inequality in the third statement of Theorem 1.3: If $R \in \mathbb{MR}$ and R is not a canonical reversible set, then $|R| \leq (k + 1)(k + 2)/2 - n(n - 1)/2 + 1$. To see this, let $\{x_1, x_2, \dots, x_{k+1}\}$ be a consistent labeling of $A(R)$. As Lemma 3.2, let i be the least integer for which $|B(x_i, R)| \neq k + 2 - i$. The lemma then supplies the exact size of $B(x_j, R)$ when $1 \leq j \leq i$. When $i + 1 \leq j \leq k + 1$, the lemma only tells us that $|B(x_j, R)| \leq k + 3 - i - n$. However, from Proposition 3.1, we also have the inequality $|B(x_j, R)| \leq k + 2 - j$. It follows that

$$|R| \leq \sum_{j=1}^{i-1} (k + 2 - j) + (k + 4 - i - n) + (n - 1)(k + 3 - i - n) + \sum_{j=i+n}^{k+1} (k + 2 - j) \\ = \frac{(k+1)(k+2)}{2} - \frac{n(n-1)}{2} + 1.$$

This completes the proof of Theorem 1.3.

The next example shows that the inequality in the third statement of Theorem 1.3 is tight.

Example 3.3 Let (n, k) be a pair with $3 \leq n \leq k$. We start with the canonical reversible set $T = T(\sigma)$ consisting of all pairs associated with the h -contiguous sequence $\sigma = (a_1, a_2, \dots, a_{k+1})$. Let i be any integer with $1 \leq i \leq k + 1 - n$. Then $u_1 \leq u_i < u_{i+n} \leq u_{k+1}$ where (a_{i+n}, b_i) belongs to $\text{Inc}(A, B)$ but not to T .

A pair $(a, b) \in T$ is adjacent to (a_{i+n}, b_i) in G_n^k if and only if $u_i < a \leq b < u_{i+n}$, so the pair (a_{i+n}, b_i) has $n(n - 1)/2$ neighbors in T . If we remove these neighbors from T and then add (a_{i+n}, b_i) , we have an independent set R of size $(k + 1)(k + 2)/2 - n(n - 1)/2 + 1$. This set is easily seen to belong to \mathbb{MR} .

4 Independent, Non-reversible Sets: Part 1

In some sense, independent, reversible sets are relatively simple objects. To our taste, independent, non-reversible sets are much more interesting, and this is the first of three sections devoted to such sets.

For a pair (n, k) , recall that $\mathbb{INR}(n, k)$ denotes the family of all independent, non-reversible sets in G_n^k . In this section, we prove Lemma 1.4: $\mathbb{INR}(n, k)$ is non-empty if and only if $n \leq 2k$.

We begin by reminding readers that independent, non-reversible sets in G_n^k contain alternating cycles. In fact, they contain strict alternating cycles. However, since they are independent, any alternating cycle they contain has size at least 3. The following straightforward proposition is essentially a restatement of the basic properties of a strict alternating cycle.

Proposition 4.1 *Let S be a non-reversible set in G_n^k . Then let $m \geq 2$ and let $C = \{(x_\alpha, y_\alpha) : \alpha \in [m]\}$ be an alternating cycle contained in S . Then for each $\alpha \in [m]$, we have $x_\alpha \preceq y_\alpha < x_{\alpha+1}$. Furthermore, if S is strict, there is no $v \in A(C) \cup B(C)$ with $y_\alpha < v < x_{\alpha+1}$.*

As a result, when $m \geq 2$ and $C = \{(x_\alpha, y_\alpha) : \alpha \in [m]\}$ is a strict alternating cycle, there are $(m - 1)!$ different orders in which the elements of $A(C) \cup B(C)$ can appear on the circle. For example, when $m = 7$, one of these possibilities is:

$$x_1 \preceq y_3 < x_4 \preceq y_6 < x_7 \preceq y_4 < x_5 \preceq y_2 < x_3 \preceq y_1 < x_2 \preceq y_5 < x_6 \preceq y_7.$$

Furthermore, for every such arrangement, there is a strict alternating cycle of this form in $\text{Inc}(A, B)$, when k is sufficiently large relative to n .

Lemma 4.2 *Let $m \geq 2$ and let $C = \{(x_\alpha, y_\alpha) : \alpha \in [m]\}$ be a strict alternating cycle in $\text{Inc}(A, B)$. Then $mn \leq 2(n + k)$.*

Proof The conclusion of the lemma holds trivially if $m = 2$, so we assume $m \geq 3$. For each $\alpha \in [m]$, let D_α be the down set of y_α in S_n^k . If $y_\alpha = b_j$, then $D_\alpha = \{a_{j-1}, a_{j-2}, \dots, a_{j-n+1}\}$ which is a set of size $n - 1$.

Since C is a strict alternating cycle, it follows that for each $a \in A(C)$, there is exactly one $\alpha \in [m]$ with $a \in D_\alpha$. Provided each element $a \in A - A(C)$ belongs to at most two of the sets in $\{D_\alpha : \alpha \in [m]\}$, then $m(n - 1) = \sum_{\alpha \in [m]} |D_\alpha| \leq 2(n + k - m) + m$ which implies $mn \leq 2(n + k)$ as claimed.

It remains to see that each $a \in A - A(C)$ belongs to at most two downsets in $\{D_\alpha : \alpha \in [m]\}$. Suppose to the contrary that there is some $a \in A - A(C)$ such that $a \in D_\alpha \cap D_\beta \cap D_\gamma$ for three distinct elements $\alpha, \beta, \gamma \in [m]$. Without loss of generality, we may assume that $y_\alpha < y_\beta < y_\gamma < a$. Since D_α, D_β and D_γ have the same size, $D_\beta \subseteq D_\alpha \cup D_\gamma$. Recall $x_{\beta+1} < y_\beta$ in the cycle C , so $x_{\beta+1} \in D_\beta$. Therefore, $x_{\beta+1} \in D_\alpha$ or $x_{\beta+1} \in D_\gamma$, which implies $x_{\beta+1} < y_\alpha$ or $x_{\beta+1} < y_\gamma$, contradicting the assumption that C was a strict alternating cycle. So, each element $a \in A - A(C)$ belongs to at most two of the sets in $\{D_\alpha : \alpha \in [m]\}$, as needed to complete the proof. \square

We are now ready to prove the first half of Lemma 1.4: If $\mathbb{INR} \neq \emptyset$, then $n \leq 2k$. Let $S \in \mathbb{INR}$. Then S contains a strict alternating cycle C of size m for some $m \geq 3$. Lemma 4.2 implies $3n \leq mn \leq 2(n + k)$, and therefore $n \leq 2k$.

For the converse, we simply show that when $n \leq 2k$, there is a strict alternating cycle of size 3 in $\text{Inc}(A, B)$. Two suitable examples are provided here.

Example 4.3 *If $n \leq k$, then*

$$C = \{(a_1, b_1), (a_2, b_{k+1}), (a_{k+2}, b_{k+2})\}$$

is a strict alternating cycle of size 3 in G_n^k . Accordingly, $C \in \mathbb{INR}(n, k)$.

Example 4.4 *If $k < n \leq 2k$, then*

$$C = \{(a_1, b_{2k+1-n}), (a_{k+1}, b_{k+1}), (a_{2k+1}, b_{2k+1})\}$$

is a strict alternating cycle of size 3 in G_n^k . Accordingly, $C \in \mathbb{INR}(n, k)$.

In time, it will become clear why these last two examples are presented in terms of the separate ranges: (i) $n \leq k$ and (ii) $k < n \leq 2k$. We also alert readers that the strict alternating cycles in these examples will resurface later in this paper.

Even though there can be arbitrarily complex strict alternating cycles in $\text{Inc}(A, B)$ when k is sufficiently large relative to n , the following lemma asserts that there is always one of small size in a maximal independent, non-reversible set.

Lemma 4.5 *If $S \in \text{MINR}(n, k)$, then S contains a strict alternating cycle of size 3.*

Proof As S is non-reversible, we can choose a strict alternating cycle $C = \{(x_\alpha, y_\alpha) : \alpha \in [m]\}$ in S of smallest size. Of course, $m \geq 3$ since S is independent. Toward a contradiction, suppose $m \geq 4$. Then $(x_1, y_2) \in \text{Inc}(A, B)$. If $(x_1, y_2) \in S$, then we may delete (x_1, y_1) and (x_2, y_2) from C and add (x_1, y_2) to obtain a strict alternating cycle of size $m - 1$ contained in S . The contradiction shows that $(x_1, y_2) \notin S$.

Since S is maximal, it follows that there is some $(a, b) \in S$ with (a, b) adjacent to (x_1, y_2) in G_n^k . Therefore, $a < y_2$ and $x_1 < b$. Thus $C' = \{(x_1, y_1), (x_2, y_2), (a, b)\}$ is a strict alternating cycle in S of size 3, another contradiction. \square

As this detail is essential to future arguments, we note that when $C = \{(x_\alpha, y_\alpha) : \alpha \in [3]\}$ is a strict alternating cycle of size 3, there are only two different ways the points of $A(C) \cup B(C)$ can appear:

- (1) $x_1 \leq y_1 < x_2 \leq y_2 < x_3 \leq y_3$ or
- (2) $x_1 \leq y_2 < x_3 \leq y_1 < x_2 \leq y_3$.

Considering our conventions about placing points of S_n^k on a circle, it is natural to say that C satisfies the *Disjoint Property* when the first of these two orders holds. Similarly, C satisfies the *Overlap Property* when the second order holds. Clearly, if C is a strict alternating cycle of size 3, then C satisfies the Disjoint Property if and only if its image under the automorphism ϕ (see Section 2.3) satisfies the Overlap Property.

Strict alternating cycles of size 3 play a vital role in the arguments to follow. Accordingly, let \mathbb{C}_{D3} denote the family of all strict alternating cycles of size 3 in $\text{Inc}(A, B)$ that satisfy the Disjoint Property while \mathbb{C}_{O3} consists of those with the Overlap Property.

For a fixed pair (n, k) with $\text{INR} \neq \emptyset$, we use the notation MINR_{D3} for the family of sets in MINR which contain a strict alternating cycle C from \mathbb{C}_{D3} . The family MSINR_{D3} is defined similarly. Analogously, define the families MINR_{O3} and MSINR_{O3} in terms of the Overlap Property. In view of our remarks about the automorphism ϕ , we state the following elementary proposition for emphasis.

Proposition 4.6 *Let (n, k) be a pair for which $\text{INR} \neq \emptyset$. Then*

- (1) $\text{MINR}_{O3} = \{\phi(S) : S \in \text{MINR}_{D3}\}$ and $\text{MINR} = \text{MINR}_{D3} \cup \text{MINR}_{O3}$;
- (2) $\text{MSINR}_{O3} = \{\phi(S) : S \in \text{MSINR}_{D3}\}$ and $\text{MSINR} = \text{MSINR}_{D3} \cup \text{MSINR}_{O3}$.

In carrying out the research for this paper, we found it more intuitive to work with the families MINR_{D3} and MSINR_{D3} , but as reflected by Proposition 4.6, if we understand these families, we really understand MINR and MSINR .

4.1 The Contraction and Expansion Lemmas

Fix a pair (n, k) and recall \mathbb{I} is the family of all independent sets in G_n^k . Let $S \in \mathbb{I}$ and $i \in [n+k]$. An ordered pair $((a, b), (x, y))$ of critical pairs belonging to S is called a *contraction*

blocking pair at i if $a = a_i, y = b_i$ and $x < b$ in S_n^k . To explain this terminology, we observe that both (a_{i+1}, b) and (x, b_{i-1}) are in $\text{Inc}(A, B)$, and they represent “contractions” of the pairs (a, b) and (x, y) , respectively. However, S does not contain either as (a_{i+1}, b) is adjacent to (x, y) and (x, b_{i-1}) is adjacent to (a, b) .

We then let $\text{FCBP}(i, S)$, for “first in a contraction blocking pair” at i , denote the set of all $(a, b) \in S$ for which there is some $(x, y) \in S$ such that $((a, b), (x, y))$ is a contraction blocking pair at i . Also, we let $\text{LCBP}(i, S)$, for “last in a contraction blocking pair” at i , denote the set of all $(x, y) \in S$ for which there is some $(a, b) \in S$ such that $((a, b), (x, y))$ is a contraction blocking pair at i . It follows from the definition that $\text{FCBP}(i, S) \neq \emptyset$ if and only if $\text{LCBP}(i, S) \neq \emptyset$.

We need two more definitions that transform an independent set via a contraction. In particular, the notation DFCL is an abbreviation for “delete first, contract last,” while DLCF is an abbreviation for “delete last, contract first.” Set

$$\begin{aligned} \text{DFCL}(i, S) &:= (S - \text{FCBP}(i, S)) \cup \{(x, b_{i-1}) : (x, y) \in \text{LCBP}(i, S)\}, \\ \text{DLCF}(i, S) &:= (S - \text{LCBP}(i, S)) \cup \{(a_{i+1}, b) : (a, b) \in \text{FCBP}(i, S)\}. \end{aligned}$$

Lemma 4.7 (Contraction Lemma) *Fix (n, k) . Let $S \in \mathbb{I}$ and $i \in [n + k]$.*

- (1) $\text{DFCL}(i, S) \in \mathbb{I}$.
- (2) $\text{DLCF}(i, S) \in \mathbb{I}$.
- (3) $|\text{DFCL}(i, S)| + |\text{DLCF}(i, S)| = 2|S|$.

Proof The conclusions of the lemma hold trivially if $\text{FCBP}(i, S)$ and $\text{LCBP}(i, S)$ are empty. Now suppose that both of these sets are non-empty. We show that $\text{DFCL}(i, S) \in \mathbb{I}$, noting that the argument for showing $\text{DLCF}(i, S) \in \mathbb{I}$ is symmetric.

The subset $S \cap \text{DFCL}(i, S)$ is independent because S is independent. Also, no two elements of $\text{DFCL}(i, S) - S$ can be adjacent since they all end at b_{i-1} . Toward a contradiction, suppose there is some $(z, w) \in S \cap \text{DFCL}(i, S)$ and some $(x, b_{i-1}) \in \text{DFCL}(i, S) - S$ with (z, w) adjacent to (x, b_{i-1}) . This requires $x < w$ and $z < b_{i-1}$ in S_n^k . Since $(x, b_i) \in S$, we know that (z, w) is not adjacent to (x, b_i) . This forces $z = a_i$. As $x < w$, then $((z, w), (x, b_i))$ is a contraction blocking pair at i so that $(z, w) \in \text{FCBP}(i, S)$ and $(z, w) \notin \text{DFCL}(i, S)$, a contradiction. Thus $\text{DFCL}(i, S) \in \mathbb{I}$.

From their definitions, it follows that

$$\begin{aligned} |\text{DFCL}(i, S)| &= |S| - |\text{FCBP}(i, S)| + |\text{LCBP}(i, S)| \quad \text{and} \\ |\text{DLCF}(i, S)| &= |S| - |\text{LCBP}(i, S)| + |\text{FCBP}(i, S)|. \end{aligned}$$

Adding these two identities yields $|\text{DFCL}(i, S)| + |\text{DLCF}(i, S)| = 2|S|$. □

Similar to contraction blocking pairs, we also have expansion blocking pairs. An ordered pair $((a, b), (x, y))$ of critical pairs belonging to S is called an *expansion blocking pair* at i when $a = a_i, y = b_{i+k}$, and $x < b$ in S_n^k . Note that S does not contain either (a_{i-1}, b) or (x, b_{i+k+1}) .

Let $\text{FEBP}(i, S)$ denote the set of all $(a, b) \in S$ for which there is some $(x, y) \in S$ such that $((a, b), (x, y))$ is an expansion blocking pair at i . Also, let $\text{LEBP}(i, S)$ denote the set of all $(x, y) \in S$ for which there is some (a, b) in S such that $((a, b), (x, y))$ is an expansion blocking pair at i . As before, $\text{FEBP}(i, S) \neq \emptyset$ if and only if $\text{LEBP}(i, S) \neq \emptyset$.

As with contractions, we define two sets which transform an independent set via an expansion: “delete first, expand last” and “delete last, expand first.” Set

$$\text{DFEL}(i, S) := (S - \text{FEBP}(i, S)) \cup \{(x, b_{i+k+1}) : (x, b_{i+k}) \in \text{LEBP}(i, S)\},$$

$$\text{DLEF}(i, S) := (S - \text{LEBP}(i, S)) \cup \{(a_{i-1}, b) : (a_i, b) \in \text{FEBP}(i, S)\}.$$

The proof of the following lemma is essentially the same as for Lemma 4.7.

Lemma 4.8 (Expansion Lemma) *Fix (n, k) . Let $S \in \mathbb{I}(n, k)$ and $i \in [n + k]$.*

- (1) $\text{DFEL}(i, S) \in \mathbb{I}$.
- (2) $\text{DLEF}(i, S) \in \mathbb{I}$.
- (3) $|\text{DFEL}(i, S)| + |\text{DLEF}(i, S)| = 2|S|$.

As this detail will be important later, we note that if $|\text{FCBP}(i, S)| \neq |\text{LCBP}(i, S)|$, then one of $|\text{DFCL}(i, S)|$ and $|\text{DLCF}(i, S)|$ is larger than $|S|$, while the other is smaller. An analogous statement holds for expansions. Furthermore, if $S \in \text{MIS}$, then $\text{DFCL}(i, S)$, $\text{DLCF}(i, S)$, $\text{DFEL}(i, S)$, and $\text{DLEF}(i, S)$ are in MIS .

While the transformations in the contraction and expansion lemmas preserve membership in MIS , they may not preserve membership in MI , as the following two examples show. Also, they may transform a non-reversible set into a reversible set. We encourage the reader to work through the claims of these two examples in preparation for the arguments to follow in the next two sections.

Example 4.9 *Let (n, k) be a pair with $k < n \leq 2k$. Recall the following strict alternating cycle, first introduced in Example 4.3:*

$$C = \{(a_1, b_{2k+1-n}), (a_{k+1}, b_{k+1}), (a_{2k+1}, b_{2k+1})\}.$$

Set

$$S := \{(x, y) : a \leq x \leq y \leq b \text{ for some } (a, b) \in C\}.$$

Then the following statements hold.

- (1) $S \in \text{MINR}(n, k)$ and $|S| = 2 + (2k + 2 - n)(2k + 1 - n)/2$.
- (2) $\text{FEBP}(1, S) = \{(a_1, y) : b_1 \leq y \leq b_{2k+1-n}\}$ so $|\text{FEBP}(1, S)| = 2k + 1 - n$.
- (3) $\text{LEBP}(1, S) = \{(a_{k+1}, b_{k+1})\}$ so $|\text{LEBP}(1, S)| = 1$.
- (4) When $n < 2k$, $\text{DFEL}(1, S)$ is a maximal non-reversible set which is smaller than S . Also, $\text{DLEF}(1, S)$ is a non-maximal reversible set.
- (5) When $n = 2k$, both $\text{DFEL}(1, S)$ and $\text{DLEF}(1, S)$ are non-maximal reversible sets.

Example 4.10 *Let (n, k) be a pair with $n \leq k$. Set*

$$S := \{(x, y) : a \leq x \leq y \leq b \text{ for some } (a, b) \in \text{Max}(S)\},$$

where

$$\text{Max}(S) = \{(a_2, b_{k+1}), (a_1, b_{k+2-n}), (a_{k+2}, b_{k+2})\}.$$

The set S is non-reversible as it contains the strict alternating cycle

$$C^* = \{(a_1, b_1), (a_2, b_{k+1}), (a_{k+2}, b_{k+2})\},$$

first introduced in Example 4.3. Then the following statements hold.

- (1) $S \in \text{MINR}(n, k)$ and $|S| = (k + 1)(k + 2)/2 + 2 - n$.
- (2) $\text{FEBP}(2, S) = \{(a_2, w) : b_{k+3-n} \leq w \leq b_{k+1}\}$ so $|\text{FEBP}(1, S)| = n - 1$.

- (3) $\text{LEBP}(2, S) = \{(a_{k+2}, b_{k+2}) \text{ so } |\text{LEBP}(k + 2, S)| = 1.$
- (4) *The set $\text{DLEF}(2, S)$ is the canonical reversible set consisting of all $(x, y) \in \text{Inc}(A, B)$ with $a_1 \leq x \leq y \leq b_{k+1}$. In particular, $\text{DLEF}(2, S)$ is the canonical reversible set associated with the h -contiguous sequence $(a_1, a_2, \dots, a_{k+1})$.*

The notions of a contraction blocking pair and an expansion blocking pair are dual in the natural symmetry on the crown, i.e., if ϕ is the automorphism of the crown S_n^k discussed in Section 2.3, then $((a, b), (x, y))$ is a contraction blocking pair at i in an independent set S if and only if $((\phi(a), \phi(b)), (\phi(x), \phi(y)))$ is an expansion blocking pair at $-i$ in $\phi(S)$. Accordingly, the pitfalls identified for expansions in the preceding examples can also occur for contractions.

5 Independent, Non-Reversible Sets: $k < n \leq 2k$

This is the second of three sections devoted to the study of independent, non-reversible sets. However, in this section, we only consider pairs (n, k) with $k < n \leq 2k$ and prove the inequality in Theorem 1.5: when $k < n \leq 2k$, if $S \in \text{INR}(n, k)$, then $|S| \leq 2 + (2k - n + 2)(2k - n + 1)/2$. In fact, we do much more. We completely determine the family $\text{MINR}(n, k)$ of all maximal independent, non-reversible sets. Consequently, we know all their possible sizes. Theorem 1.5 simply extracts the largest value among them.

For the remainder of this section, fix a pair (n, k) with $k < n \leq 2k$ and use the abbreviations $\text{MINR} = \text{MINR}(n, k)$, $\text{MSINR} = \text{MSINR}(n, k)$ and $\text{INR} = \text{INR}(n, k)$.

From Lemma 4.5, we know that every set $S \in \text{MINR}$ contains a strict alternating cycle of size 3. In the range $k < n \leq 2k$, we can say more.

Proposition 5.1 *If $S \in \text{INR}$, then every strict alternating cycle contained in S has size 3.*

Proof Fix a set $S \in \text{INR}$ and a strict alternating cycle C of size m in S . Since S is independent, $m \neq 2$. Toward a contradiction, suppose $m \geq 4$. By applying Lemma 4.2 to the pair (S, C) , we conclude that $4n \leq mn \leq 2(n + k)$. This implies $n \leq k$, a contradiction. \square

From Proposition 4.5, we know that $\text{MINR} = \text{MINR}_{D3} \cup \text{MINR}_{O3}$. Furthermore, Proposition 4.6 implies $\text{MINR}_{O3} = \{\phi(S) : S \in \text{MINR}_{D3}\}$, so both MINR_{D3} and MINR_{O3} are non-empty when $n \leq 2k$. However, it is not clear from their definitions that MINR_{D3} and MINR_{O3} are disjoint. This is a detail we will discover.

Recall that for any $(a, b), (x, y) \in \text{Inc}(A, B)$, we say (x, y) is contained in (a, b) provided $a \leq x \leq y \leq b$. The result is an inclusion order on $\text{Inc}(A, B)$. In the descriptions to follow, we will reference down sets and up sets in this poset on $\text{Inc}(A, B)$.

We show that every set $S \in \text{MINR}_{D3}$ is a down set in $\text{Inc}(A, B)$. Using duality, every set in MINR_{O3} is an up set in $\text{Inc}(A, B)$. In contrast, there are canonical reversible sets which are neither up sets nor down sets in $\text{Inc}(A, B)$. To see this, consider the crown S_4^5 and the canonical reversible set in Example 2.5. The set contains (a_7, b_9) but does not contain either (a_7, b_8) or (a_6, b_9) . In the next section, when $n \leq k$, the analysis of maximal, non-reversible sets is considerably more complex as they too may be neither down sets nor up sets in $\text{Inc}(A, B)$.

If we fully understand either of the two subfamilies MINR_{D3} and MINR_{O3} , then we have all information for MINR . So, for the remainder of this subsection, we focus on MINR_{D3} and emphasize our restriction to the case $k < n \leq 2k$.

At several steps in the discussion to follow, we will need the next proposition.

Proposition 5.2 *There do not exist pairs (v_1, v_2) and (v_3, v_4) of points on the circle, each of size at most $k + 1$, so that (1) $v_1 \preceq v_3 \preceq v_2$ and $v_1 \preceq v_4 \preceq v_2$ and (2) for each $i \in [n + k]$, u_i is in (v_1, v_2) or u_i is in (v_3, v_4) .*

Proof Suppose to the contrary that two such pairs exist. By (1), (v_1, v_2) and (v_3, v_4) share at least 2 points. Since each has size at most $k + 1$, (2) yields the inequality $n + k \leq 2(k + 1) - 2 = 2k$. However, this implies $n \leq k$, a contradiction. \square

Next, we give a construction for sets belonging to MINR_{D_3} . Fix a positive integer t and let $C_0 = \{(x_\alpha, y_\alpha) : \alpha \in [2t + 1]\}$ be an alternating cycle (in general, not strict) in $\text{Inc}(A, B)$ such that the following two conditions are satisfied:

- (1) $x_1 \preceq y_1 \prec x_2 \preceq y_2 \prec \dots \prec x_{2t+1} \preceq y_{2t+1} \prec x_1$; and
- (2) for each $\alpha \in [2t + 1]$, the size of $(x_\alpha, y_{\alpha+t})$ is $k + 1$.

In the discussion to follow, we refer to these two conditions as the *Matching Conditions*. Given an alternating cycle C_0 satisfying the Matching Conditions, we then let $D(C_0)$ be the down set of C_0 , i.e. all pairs $(x, y) \in \text{Inc}(A, B)$ for which there is an $\alpha \in [2t + 1]$ with $x_\alpha \preceq x \preceq y \preceq y_\alpha$. If we let s_α denote the size of (x_α, y_α) , then $|D(C_0)| = \sum_{\alpha=1}^{2t+1} s_\alpha(s_\alpha + 1)/2$.

Example 5.3 *When $n = 47$ and $k = 42$, the following incomparable pairs form an alternating cycle C_0 of size 7 satisfying the Matching Conditions for $t = 3$:*

$$C_0 = \{(a_1, b_4), (a_{13}, b_{20}), (a_{25}, b_{31}), (a_{37}, b_{43}), (a_{51}, b_{55}), (a_{67}, b_{67}), (a_{78}, b_{79})\}.$$

It follows that

$$|D(C_0)| = \binom{5}{2} + \binom{9}{2} + \binom{8}{2} + \binom{8}{2} + \binom{6}{2} + \binom{2}{2} + \binom{3}{2}.$$

These next two exercises are left for the reader. For the first, one may consider the $2t + 1$ pairs $(y_{\alpha+t+1}, x_\alpha)$ of length n with the property that, for any point u on the circle, there are at most $t + 1$ values α for which $y_{\alpha+t+1} \preceq u \preceq x_\alpha$.

Exercise 5.4 If $k < n \leq 2k$ and t is a positive integer, then there is an alternating cycle C_0 in $\text{Inc}(A, B)$ of size $2t + 1$ which satisfies the Matching Conditions if and only if $t(n - k) \leq k$.

Exercise 5.5 If $k < n \leq 2k$, $t(n - k) \leq k$, and $(s_1, s_2, \dots, s_{2t+1})$ is a sequence of positive integers, then there is an alternating cycle $C_0 = \{(x_\alpha, y_\alpha) : \alpha \in [2t + 1]\}$ satisfying the Matching Conditions with the size of (x_α, y_α) equal to s_α for each $\alpha \in [2t + 1]$ if and only if $\sum_{\alpha=1}^{2t+1} s_\alpha = k + 2t + 1 - t(n - k)$.

As an illustration of the preceding exercise, we note that in Example 5.3, $n = 47$ and $k = 42$ so that the maximum value of t is 8. We chose $t = 3$ in which case the sum of the sizes of the seven pairs in C_0 was $42 + 6 + 1 - 3 \cdot 5 = 34$. Note that when $n = 2k$, the maximum value of t is 1 and all pairs must have size 1 as illustrated in Example 4.4.

Lemma 5.6 *Let $t \geq 1$ and let $C_0 = \{(x_\alpha, y_\alpha) : \alpha \in [2t + 1]\}$ be an alternating cycle satisfying the Matching Conditions. Then the down set $D(C_0)$ is in MINR_{D_3} .*

Proof Any two distinct pairs (x, y) and (x', y') in $D(C_0)$ are non-adjacent since there is some $\alpha \in [2t + 1]$ so that both (x, y) and (x', y') are contained in $(x_\alpha, y_{\alpha+t})$ which has size $k + 1$. Also, S is non-reversible as it contains the alternating cycle C_0 . So $D(C_0) \in \text{INR}$. Since there is a subset of C_0 which constitutes a strict alternating cycle of size 3 by Proposition 5.1. Further, it is clear that this strict alternating cycle satisfies the Disjoint Property.

To complete the proof, we need only show that $D(C_0)$ is a maximal independent set. Let (a, b) be a pair in $\text{Inc}(A, B)$ which does not belong to $D(C_0)$. We show there is some (x, y) in $D(C_0)$ which is adjacent to (a, b) in G_n^k .

Suppose first that there is some $\alpha \in [2t + 1]$ with $x_\alpha \leq a \leq y_\alpha$. Since $(a, b) \notin D(C_0)$, we have $a \leq y_\alpha < b$. If $a \leq y_\alpha < x_{\alpha+t+1} \leq b$, then (a, b) and $(x_{\alpha+t+1}, y_\alpha)$ violate Proposition 5.2. If $a \leq y_\alpha < b < x_{\alpha+t+1}$, then (a, b) is adjacent to $(x_{\alpha+t+1}, y_{\alpha+t+1})$.

So we may assume that there is some α where $y_\alpha < a < x_{\alpha+1}$. If $a \leq b < x_{\alpha+t+1}$, then (a, b) is adjacent to $(x_{\alpha+t+1}, y_{\alpha+t+1})$. If $x_{\alpha+t+1} \leq b < x_\alpha$, then (a, b) is adjacent to (x_α, y_α) . These observations complete the proof of the lemma. \square

The next result asserts that the construction we have just presented actually defines the family MINR_{D_3} .

Theorem 5.7 *If $k < n \leq 2k$ and $S \in \text{MINR}_{D_3}$, then there is some $t \geq 1$ and an alternating cycle C_0 of size $2t + 1$ which satisfies the Matching Conditions and has the property that $S = D(C_0)$.*

Proof Fix $S \in \text{MINR}_{D_3}$. The *span* of a strict alternating cycle is the sum of the sizes of the incomparable pairs which define the cycle. Of all the strict alternating cycles of size 3 contained in S which satisfy the Disjoint Property, we choose one with maximum span. Call this cycle $C := \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$. We now establish three claims needed to complete the proof of Theorem 5.7.

Claim *For each $\alpha \in \{1, 2, 3\}$, the following two statements hold:*

- (1) $x < y_\alpha$ in S_n^k for all x with $y_\alpha < x \leq y_{\alpha+1}$; and
- (2) $x_\alpha < y$ in S_n^k for all y with $x_{\alpha+2} \leq y < x_\alpha$.

Proof Fix x with $y_\alpha < x \leq y_{\alpha+1}$ and suppose, toward a contradiction, that $(x, y_\alpha) \in \text{Inc}(A, B)$. Since $(x_\alpha, y_{\alpha+1}) \in \text{Inc}(A, B)$, both (x, y_α) and $(x_\alpha, y_{\alpha+1})$ have size at most $k + 1$. Because C satisfies the Disjoint Property, we have $x_\alpha \leq y_\alpha < y_{\alpha+1}$. As a result, (x, y_α) and $(x_\alpha, y_{\alpha+1})$ violate Proposition 5.2.

For the second statement, suppose $(x_\alpha, y) \in \text{Inc}(A, B)$. A similar contradiction is reached by considering the pairs (x_α, y) and $(x_{\alpha+2}, y_\alpha)$. \square

Considering S as a subposet of $\text{Inc}(A, B)$ ordered by inclusion, let $\text{Max}(S)$ denote the maximal elements of S .

Claim *The pairs in C belong to $\text{Max}(S)$. Furthermore, if (x, y) and (x', y') are distinct pairs in $\text{Max}(S)$, then they are disjoint.*

Proof We first show that any pair $(x, y) \in \text{Max}(S)$ overlaps at most one of the pairs in C . If $(x, y) \in \text{Max}(S)$ and (x, y) overlaps all three pairs in C , then there is some α for which

$(y_\alpha, x_{\alpha+2})$ is contained in (x, y) . However, $(x_{\alpha+2}, y_\alpha)$ and (x, y) then violate Proposition 5.2.

Now suppose that there is a pair $(x, y) \in \text{Max}(S)$ that overlaps (x_α, y_α) and $(x_{\alpha+1}, y_{\alpha+1})$ but not $(x_{\alpha+2}, y_{\alpha+2})$. Using the previous claim, this implies that (x, y) is adjacent to $(x_{\alpha+2}, y_{\alpha+2})$ in G_n^k . The contradiction completes the proof that a pair $(x, y) \in \text{Max}(S)$ overlaps at most one pair in C .

Now suppose that some (x_α, y_α) is not in $\text{Max}(S)$. Then there is a pair $(x, y) \in \text{Max}(S)$ so that (x_α, y_α) is properly contained in (x, y) . Because (x, y) does not overlap any other pairs in C , it follows that we can replace (x_α, y_α) in C with (x, y) and obtain a strict alternating cycle C' whose span is larger than the span of C . The contradiction shows that the pairs of C belong to $\text{Max}(S)$.

We now show that all pairs in $\text{Max}(S)$ are disjoint. To do this, we first show that no pair in $\text{Max}(S) - C$ overlaps a pair in C . Toward a contradiction, suppose $(x, y) \in \text{Max}(S) - C$ overlaps $(x_1, y_1) \in C$. Thus far we know that either $x_1 < x \leq y_1 < y$ or $x < x_1 \leq y < y_1$. We supply the argument only for the first, as a similar proof holds for the second. Because (x, y) overlaps at most one pair in C , we have $x_1 < x \leq y_1 < y < x_2 \leq y_2$. Since the size of (x_1, y) is less than the size of incomparable pair (x_1, y_2) , we have $(x_1, y) \in \text{Inc}(A, B)$. However, (x, y) and (x_1, y_1) are both in $\text{Max}(S)$ so $(x_1, y) \notin S$. Since S is a maximal independent set, it follows that there is some $(a, b) \in S$ with (a, b) adjacent to (x_1, y) in G_n^k . Then we must have $(a, y_1) \in \text{Inc}(A, B)$; otherwise (a, b) is adjacent to (x_1, y_1) in G_n^k . Similarly, we must have $(x, b) \in \text{Inc}(A, B)$; otherwise (a, b) is adjacent to (x, y) . As a result, the intervals (a, y_1) and (x, b) violate Proposition 5.2, a contradiction. Therefore, the intervals in $\text{Max}(S) - C$ are disjoint from the intervals in C .

Finally, we consider two distinct pairs (x, y) and (x', y') in $\text{Max}(S) - C$. Toward a contradiction, suppose that they overlap with $x < x' \leq y < y'$. We have already established that neither overlaps a pair in C . So we may assume $x_1 \leq y_1 < x < x' \leq y < y' < x_2 \leq y_2$. Because $(x_1, y_2) \in \text{Inc}(A, B)$, we may conclude $(x, y') \in \text{Inc}(A, B)$. Furthermore, $(x, y') \notin S$ because $(x, y) \in \text{Max}(S)$. As before, this implies there exists $(a, b) \in S$ with (x, y') adjacent to (a, b) in G_n^k . The same argument yields a violation of Proposition 5.2, a contradiction. Thus all pairs in $\text{Max}(S)$ are disjoint, completing the proof of the claim. \square

Claim *The pairs in S form a downset in $\text{Inc}(A, B)$.*

Proof Let $(x, y) \in \text{Max}(S)$ and let (x', y') be any pair from $\text{Inc}(A, B)$ contained in (x, y) . Toward a contradiction, suppose (x', y') is not in S . Then there is some $(a, b) \in S$ with (a, b) adjacent to (x', y') in G_n^k .

Choose a maximal element $(z, w) \in \text{Max}(S)$ with $(a, b) \subseteq (z, w)$. Since $(x, y) \in \text{Inc}(A, B)$ while (a, b) and (x', y') are adjacent, we do not have $(z, w) = (x, y)$. So (z, w) and (x, y) are disjoint by the previous claim. This, together with the fact that $x' < b$ and $a < y'$, implies that (x, y) and (z, w) are adjacent, a contradiction to the fact that S is an independent set. \square

Let $m = |\text{Max}(S)|$ and label the elements of $\text{Max}(S)$ as $\{(z_\beta, w_\beta) : 1 \leq \beta \leq m\}$ so that

$$z_1 \leq w_1 < z_2 \leq w_2 < z_3 \leq w_3 < \dots < z_m \leq w_m < z_1.$$

Let $\alpha \in [m]$ and let $(z_\alpha, w_\alpha) = (a_i, b_j)$. Then we must have $(a_i, b_j) \in \text{FEBP}(i, S) \cap \text{LEBP}(j - k, S)$. It follows immediately that there are distinct integers $\beta, \gamma \in [m]$ so that $((z_\alpha, w_\alpha), (z_\beta, w_\beta))$ is an expansion blocking pair at i and $((z_\gamma, w_\gamma), (z_\alpha, w_\alpha))$ is an

expansion blocking pair at $j - k$. Furthermore, we may conclude that $\gamma = \beta + 1$ since otherwise $(z_{\beta+1}, w_{\beta+1})$ is adjacent to (z_α, w_α) . Each pair in $\text{Max}(S)$ participates in two blocking pairs like this, so we may deduce that there is some t for which $\beta = \alpha + t$ and $\gamma + t = \alpha$ wherein $m = 2t + 1$ and the size of $(z_\alpha, w_{\alpha+t})$ is $k + 1$. This implies that $C_0 := \text{Max}(S)$ is indeed a cycle which satisfies the Matching Conditions and $S = D(C_0)$. These observations complete the proof of Theorem 5.7. \square

Having established the form of all sets in MINR_{D_3} , we proceed with the task of analyzing the possible sizes. Let t be a positive integer and let $C_0 = \{(x_\alpha, y_\alpha) : \alpha \in [2t + 1]\}$ be an alternating cycle of size $2t + 1$ satisfying the Matching Conditions. As we have already noted,

$$|D(C_0)| = \sum_{\alpha=1}^{2t+1} \frac{(s_\alpha + 1)s_\alpha}{2}.$$

Because Exercise 5.5 shows $\sum_{\alpha=1}^{2t+1} s_\alpha = k + 2t + 1 - t(n - k)$, a simple calculation shows that $|D(C_0)|$ is maximized when there is at most one value of s_α that is larger than 1. Accordingly, when t is fixed, the maximum value of $|D(C_0)|$ is attained when (1) $x_\alpha = a_{\alpha k+1}$ and $y_\alpha = b_{\alpha k+1}$ for all $\alpha \in [2t]$ and (2) $x_{2t+1} = a_1$ and $y_{2t+1} = b_{k+1-t(n-k)}$. (Note that $1 \leq k + 1 - t(n - k) \leq k$.) With these values, $s_\alpha = 1$ for all $\alpha \in [2t]$, while $s_{2t+1} = k + 1 - t(n - k)$. This yields $|D(C_0)| = 2t + (k + 1 - t(n - k))(k + 2 - t(n - k))/2$ which is maximized when $t = 1$, resulting in the value $2 + (2k + 1 - n)(2k + 2 - n)/2$. With this observation, the proof of Theorem 1.5 is complete.

We comment that when $k < n \leq 2k$, there is essentially only one extremal example. To be more precise, for

$$C_0 = \{(a_1, b_{2k-n+1}), (a_{k+1}, b_{k+1}), (a_{2k+1}, b_{2k+1})\},$$

each set in MSINR is obtained from $D(C_0)$ via the natural symmetries ϕ and τ .

6 Independent, Non-Reversible Sets: $n \leq k$

This is the third and last of the sections devoted to the study of independent, non-reversible sets. Here, we will prove Theorem 1.6: if $n \leq k$ and S is an independent, non-reversible set in G_n^k , then $|S| \leq (k + 1)(k + 2)/2 + 2 - n$. With the construction given in Example 4.10, this inequality is best possible. As noted previously, we are actually able to determine the entire family $\text{MSINR}(n, k)$ of all maximum size independent, non-reversible sets, but due to space limitations, we restrict our discussion to determining the common size of the posets in this family.

We consider Theorem 1.6 to be our capstone result, with the difficulty rooted in the fact that the family $\text{MINR}(n, k)$ of maximal independent, non-reversible sets is much more complicated when $n \leq k$. To illustrate this complexity, the following anomaly shows the existence of sets in $\text{MINR}(n, k)$ which are neither up sets nor down sets in $\text{Inc}(A, B)$ as opposed to the structure found when $k < n \leq 2k$ in the previous section. (The proof of this result is left as an exercise.)

Proposition 6.1 *Let m be a positive integer and let $J \subseteq [m]$ be arbitrary. Then there are pairs (n, k) with $n \leq k$, a set $S \in \text{MINR}(n, k)$, and a set $\{(x_i, y_i) : 1 \leq i \leq m\} \subseteq \text{Inc}(A, B)$ such that (1) (x_i, y_i) is contained in (x_{i+1}, y_{i+1}) for all $i \in [m - 1]$ and (2) $(x_i, y_i) \in S$ if and only if $i \in J$.*

6.1 Further Details on Canonical Reversible Sets

Ironically, in order to prove a result about independent, non-reversible sets, we must first go back and study canonical reversible sets in greater detail.

The results of this subsection apply to any pair (n, k) . Let T be a canonical reversible set. Then there is a uniquely determined h -contiguous sequence $\sigma = (x_1, x_2, \dots, x_{k+1})$ for which $T = T(\sigma)$. We refer to x_1 as the *base* element of σ . When $i \geq 2$, an element x_i is called a *leading* element of σ when x_i is the first element in the contiguous set $\{x_1, x_2, \dots, x_i\}$. Dually, x_i is referred to as a *trailing* element of σ when x_i is the last element of the contiguous set $\{x_1, x_2, \dots, x_i\}$.

It is worth noting that that there are $(n + k)2^k$ canonical reversible sets in G_n^k since there are $n + k$ choices for the base element of σ . Then, for each $i \geq 2$, there are two choices: either x_i is leading or trailing.

Recall that for each $a_i \in A$, we have $I(a_i) = \{b_i, b_{i+1}, \dots, b_{i+k}\}$. Respecting their appearance on the circle, b_i is the first element of $I(a_i)$ and b_{i+k} is the last. Dually, a_{i-k} is the first element of $I(b_i)$ and a_i is the last. In the same sense, when $0 \leq r \leq k$, a subset of the form $\{b_i, b_{i+1}, \dots, b_{i+r}\}$ is an *initial portion* of $I(a_i)$ while $\{b_{i+r}, b_{i+r+1}, \dots, b_{i+k}\}$ is a *terminal portion*. Analogously, an initial portion of $I(b_i) = \{a_{i-k}, \dots, a_i\}$ has the form $\{a_{i-k}, \dots, a_{i-r}\}$ while a terminal portion is of the form $\{a_{i-r}, \dots, a_i\}$, where $0 \leq r \leq k$.

The following proposition is an easy exercise, following essentially from the definition of a canonical reversible set.

Proposition 6.2 *Let $\sigma = (x_1, x_2, \dots, x_{k+1})$ be an h -contiguous sequence and let $T = T(\sigma)$ be the canonical reversible set associated with σ . Then the following statements hold.*

- (1) $B(x_1, T) = I(x_1)$.
- (2) *If $i \geq 2$ and x_i is a leading element of σ , then $B(x_i, T)$ is the initial portion of $I(x_i)$ with $k + 2 - i$ elements.*
- (3) *If $i \geq 2$ and x_i is a trailing element of σ , then $B(x_i, T)$ is the terminal portion of $I(x_i)$ with $k + 2 - i$ elements.*

By symmetry, it follows that for each $y \in B(T)$, the set $A(y, T)$ is either an initial or terminal portion of $I(y)$.

Below are two elementary exercises. The first is a consequence of the order of x and x' in the associated h -contiguous sequence.

Proposition 6.3 *Let T be a canonical reversible set, and let $(x, y), (x', y') \in T$. If $x \leq y < x' \leq y'$ and $x' < y$ in S_n^k , then $(x, y') \in T$.*

Proposition 6.4 *Let T be a canonical reversible set, and let (a_i, b_j) be a pair in T . Then the following statements hold.*

- (1) *If $(a_i, b_j) \in \text{FEBP}(i, T)$, then $(a_{j+1}, b_{i+k}) \in \text{LEBP}(i, T)$.*
- (2) *If $(a_i, b_j) \in \text{LEBP}(j, T)$, then $(a_{j-k}, b_{i-1}) \in \text{FEBP}(j, T)$.*
- (3) *If $(a_i, b_j) \in \text{FCBP}(i, T)$, then $(a_{j+n-1}, b_i) \in \text{LCBP}(i, T)$.*
- (4) *If $(a_i, b_j) \in \text{LCBP}(j, T)$, then $(a_j, b_{i-n+1}) \in \text{FCBP}(j, T)$.*

The next result will be quite useful in upcoming proofs.

Lemma 6.5 *Let $C = \{(x_\alpha, y_\alpha) : \alpha \in [3]\}$ be a strict alternating cycle in $\text{Inc}(A, B)$ that satisfies the Disjoint Property. If T is a canonical reversible set containing (x_2, y_2) and (x_3, y_3) , then there are integers $m, p \in [n + k]$ so that the following statements hold.*

- (1) $A(y_2, T)$ is a terminal portion of $I(y_2)$ starting at a_m .
- (2) $B(x_3, T)$ is an initial portion of $I(x_3)$ ending at b_p .
- (3) $a_{p-k} \leq a_m \leq x_2 \leq y_2 < x_3 \leq y_3 \leq b_p \leq b_{m+k}$.
- (4) $(z, w) \in T$ whenever $a_m \leq z \leq y_2 \leq w \leq b_{m+k}$.
- (5) $(z, w) \in T$ whenever $a_{p-k} \leq z \leq x_3 \leq w \leq b_p$.

Proof First we claim that $A(y_2, T)$ is a terminal portion of $I(y_2)$. If instead $A(y_2, T)$ were an initial portion of $I(y_2)$, then $(x_1, y_2) \in T$ because x_1 is incomparable to y_2 and $x_1 < x_2 \leq y_2$. But (x_1, y_2) is adjacent to (x_3, y_3) , a contradiction. A symmetric argument shows that $B(x_3, T)$ is an initial portion of $I(x_3)$.

Choose $m, p \in [n + k]$ so that $A(y_2, T)$ starts at a_m and $B(x_3, T)$ ends at b_p . The first two statements of the lemma simply reflect these choices. Furthermore, since $(a_m, y_2) \in T$ and $(x_3, b_p) \in T$, Proposition 6.3 implies $(a_m, b_p) \in T$. The fact that a_m is incomparable to b_p yields the ordering in the third statement of the lemma.

By the choice of m , it follows that $(a_m, y_2) \in \text{FEBP}(m, T)$. If $y_2 = b_r$, then Proposition 6.4 implies $(a_{r+1}, b_{m+k}) \in \text{LEBP}(m, T)$. Clearly, $B(a_{r+1}, T)$ is an initial portion of $I(a_{r+1})$ ending at b_{m+k} because of the blocking pair. Analogously, if $x_3 = a_s$, then $A(b_{s-1}, T)$ is a terminal portion of $I(b_{s-1})$ starting at a_{p-k} .

Now, let (z, w) be a pair with $a_m \leq z \leq y_2 \leq w \leq b_{m+k}$. We claim that $(z, w) \in T$. This claim holds trivially if $w = y_2$. When $y_2 < w \leq b_{m+k}$, the claim is true by Proposition 6.3 because T contains both (z, b_r) and (a_{r+1}, w) . The fifth statement of the lemma follows from a similar argument. □

Before turning to the main body of the proof for Theorem 1.6, we require one more result that holds for any independent set. In particular, if $S \in \mathbb{I}$, then we have the trivial inequality $|B(a_1, S)| + |A(b_{k+2}, S)| \leq k + 1$. But there are circumstances in which we can sharpen this inequality.

Lemma 6.6 *Let $S \in \mathbb{I}$. If S contains pairs $(a_1, y), (x, b_{k+2})$ with $a_1 \leq y < x \leq b_{k+2}$, then $|B(a_1, S)| + |A(b_{k+2}, S)| \leq k + 3 - n$.*

Proof Define a 3-coloring of $[k + 1]$ as follows. Color i red if $(a_1, b_i) \in S$; blue if $(a_{i+1}, b_{k+2}) \in S$; and green if neither (a_1, b_i) nor (a_{i+1}, b_{k+2}) are in S . The coloring is well-defined as (a_1, b_i) and (a_{i+1}, b_{k+2}) are adjacent and cannot both belong to S .

With these definitions, it is clear that $|B(a_1, S)| + |A(b_{k+2}, S)|$ is just the number of integers in $[k + 1]$ which have been colored red or blue. In the assumption of the lemma, if $y = b_i$ and $x = a_{j+1}$, then i is colored red, j is colored blue, and $i < j$. Let m be the largest integer for which m is red and there exists $j, m < j \leq k + 1$, such that j is blue. Then let p be the least integer with $m < p$ such that p is blue. Clearly, all integers r with $m < r < p$ are green. However, it is clear that $p \geq m + n - 1$; otherwise (a_1, b_m) is adjacent to (a_{p+1}, b_{k+2}) . So there are at least $n - 2$ elements of $[k + 1]$ which are green. In turn, at most $(k + 1) - (n - 2) = k + 3 - n$ are red or blue. □

6.2 The Main Body of the Proof of Theorem 1.6

Fix a pair (n, k) with $n \leq k$. Since MSINR_{D_3} and MSINR_{O_3} are both non-empty (Lemma 1.4 and Proposition 4.6), we choose to focus on a set $S \in \text{MSINR}_{D_3}$. Toward a contradiction, assume that $|S| > (k + 1)(k + 2)/2 + 2 - n$. First we provide an outline of the proof.

Recall that MSINR_{D_3} is the family of all maximum size, independent, non-reversible sets in G_n^k that contain a strict alternating cycle of size 3 which has the Disjoint Property. For each $S \in \text{MSINR}_{D_3}$, let $\mathbb{C}_{D_3}(S)$ denote the family of strict alternating cycles of size 3 in S that have the Disjoint Property. Define \mathbb{C}_{D_3} to be the union of $\mathbb{C}_{D_3}(S)$ taken over all $S \in \text{MSINR}_{D_3}$. Analogously, for each $C \in \mathbb{C}_{D_3}$, we let $\text{MSINR}_{D_3}(C)$ denote the family of all $S \in \text{MSINR}_{D_3}$ with $C \in \mathbb{C}_{D_3}(S)$.

After establishing some tools, we show that the following strict alternating cycle (first introduced in Examples 4.3 and 4.10) is in \mathbb{C}_{D_3} :

$$C^* = \{(a_1, b_1), (a_2, b_{k+1}), (a_{k+2}, b_{k+2})\}.$$

We then find a set in $\text{MSINR}_{D_3}(C^*)$ such that, for each (x, y) in the family, both $a_1 \leq x \leq a_{k+2}$ and $b_1 \leq y \leq b_{k+2}$. Once this is accomplished, the following lemma gives the final contradiction.

Lemma 6.7 *Consider a set $S \in \text{INR}$ which contains a strict alternating cycle $\{(x_\alpha, y_\alpha) : \alpha \in [3]\}$ in \mathbb{C}_{D_3} with $x_1 = a_1$ and $y_3 = b_{k+2}$. If $A(S) \subseteq \{a_1, a_2, \dots, a_{k+2}\}$ and $B(S) \subseteq \{b_1, b_2, \dots, b_{k+2}\}$, then $|S| \leq (k + 1)(k + 2)/2 + 2 - n$.*

Proof Partition S as $M_1 \cup M_2 \cup M_3$, where M_1 consists of all $(a, b) \in S$ with $a = a_1$ or $b = b_{k+2}$, M_2 consists of all $(a, b) \in S$ with $a_2 \leq a \leq b \leq b_{k+1}$, and $M_3 = S - (M_1 \cup M_2)$. From Lemma 6.6, $|M_1| = |B(a_1, S)| + |A(b_{k+2}, S)| \leq k + 3 - n$.

To prove $|M_2| + |M_3| \leq k(k + 1)/2$, we will define a 1–1 map f which assigns to each $(a, b) \in M_3$ a pair $f(a, b)$ satisfying the following two conditions: (1) $f(a, b)$ is contained in (a_2, b_{k+1}) and (2) $f(a, b)$ is adjacent to (a, b) (so $f(a, b) \notin M_2$). Since the number of pairs in $\text{Inc}(A, B)$ which are contained in (a_2, b_{k+1}) is exactly $k(k + 1)/2$, the inequality $|M_2| + |M_3| \leq k(k + 1)/2$ follows.

If $(a, b) \in M_3$, then there are integers i, j with $1 \leq i < j \leq k + 2$ so that $(a, b) = (a_j, b_i)$ by the hypothesis of the lemma. Note $i < j - 2$ as $n \geq 3$. We then define $f(a, b) = f(a_j, b_i) = (a_{i+1}, b_{j-1})$. It is easy to see that all requirements are met by this map.

The proof of the lemma is now complete since

$$|S| = |M_1| + (|M_2| + |M_3|) \leq (k + 3 - n) + \frac{k(k + 1)}{2} = \frac{(k + 1)(k + 2)}{2} + 2 - n.$$

□

Now that we have an outline of the proof of Theorem 1.6, we begin with a lemma which gives flexibility in applying the contraction lemma (Lemma 4.7). This is the first step as we steer toward finding a set in MSINR_{D_3} which contains cycle C^* .

Lemma 6.8 *Let $S \in \text{MSINR}_{D_3}$ and $i \in [n + k]$. Under the assumption that $|S| > (k + 1)(k + 2)/2 + 2 - n$, both $\text{DFCL}(i, S)$ and $\text{DLCF}(i, S)$ belong to MSINR .*

Proof First note that if $\text{FCBP}(i, S) = \emptyset$, then $\text{DFCL}(i, S) = \text{DLCF}(i, S) = S$. If both $\text{DFCL}(i, S)$ and $\text{DLCF}(i, S)$ are non-reversible, then the fact that $|\text{DFCL}(i, S)| + |\text{DLCF}(i, S)| = 2|S|$ forces both to belong to MSINR . So if the lemma fails, one or both of $\text{DFCL}(i, S)$ and $\text{DLCF}(i, S)$ must be reversible. In this case, one of the following two statements must apply:

- (1) $\text{DFCL}(i, S)$ is reversible and $|\text{DFCL}(i, S)| \geq |S|$, or
- (2) $\text{DLCF}(i, S)$ is reversible and $|\text{DLCF}(i, S)| \geq |S|$.

We assume the first statement applies and argue to a contradiction. From the details of the argument, the proof for when the second statement holds is symmetric.

After a relabeling, we may assume $i = 1$. Let R denote the reversible set $\text{DFCL}(1, S)$ and let $C = \{(x_\alpha, y_\alpha) : \alpha \in [3]\}$ be any member of $\mathbb{C}_{D_3}(S)$. Clearly, one of the pairs in C belongs to $\text{FEBP}(1, S)$ because R is reversible. Then relabel the pairs in C so that $(x_1, y_1) \in \text{FEBP}(1, S)$ and $x_1 = a_1$.

Let $T \in \mathbb{MR}$ with $R \subseteq T$ so that $|S| \leq |R| \leq |T|$. Since $|S| > (k + 1)(k + 2)/2 + 2 - n$, it follows that $|T - R| < n - 2$. We emphasize this inequality as many of our contradictions result from this fact.

If T is not a canonical reversible set, then Theorem 1.3 implies $|T| \leq (k + 1)(k + 2)/2 - n(n - 1)/2 + 1$. As $|T| \geq |S|$, this contradicts our assumption about the size of S . It follows that T is a canonical reversible set. As T contains the pairs (x_2, y_2) and (x_3, y_3) , let $m, p \in [n + k]$ be the integers specified by Lemma 6.5. We now break our analysis into two cases based upon the location of b_p .

Case 1: Suppose $y_3 \leq b_p < b_{n+k}$. Say $y_1 = b_j$. Since C is a strict alternating cycle, we have $(x_3, y_1) \in \text{Inc}(A, B)$. Therefore $a_j < a_{j+n-1} < x_3$. We further divide this case according to the location of a_{p-k} .

First suppose $a_{p-k} \leq a_{j+1} \leq x_2$. By Lemma 6.5, it follows that T contains all $n - 1$ pairs in the following set:

$$N = \{(a_{j+\gamma}, b_p) : \gamma \in [n - 1]\}.$$

All pairs in N are adjacent to (x_1, y_1) , so $N \subset T - S$. Furthermore, since $b_p \neq b_{n+k}$, we know that $N \cap R = \emptyset$ as all pairs in $R - S$ end at b_{n+k} . However, this would imply that $|T - R| \geq n - 1$, which contradicts the inequality $|T - R| < n - 2$.

Therefore, it must be the case that $a_{j+1} < a_{p-k} \leq x_2$. Let t_3 count the number of points u_δ on the circle with $b_p < u_\delta < a_1$ and t_1 the number of points u_δ with $y_1 < u_\delta < a_{p-k}$. Because $b_p < b_{n+k} < a_1$, we have $t_3 > 0$. Similarly, $t_1 > 0$. Now let s_1 be the size of (x_1, y_1) . Then $s_1 > 0$. Since $s_1 + t_1 + t_3$ counts the number of points u_δ on the circle with $b_p < u_\delta < a_{p-k}$, we know that $s_1 + t_1 + t_3 = n - 1$.

Now consider the pairs in the following set which are contained in T :

$$N' = \{(a_m, b_{p+1-\beta}) : \beta \in [s_1 + t_1]\} \cup \{(a_{p-k-1+\gamma}, b_p) : \gamma \in [s_1 + t_3]\}.$$

Clearly, $|N'| = 2s_1 + t_1 + t_3 - 1 \geq s_1 + t_1 + t_3 = n - 1$. Furthermore, all pairs in N' are adjacent to (x_1, y_1) so $N' \subseteq T - S$. Since no pair in N' ends at b_{n+k} , we conclude that $N' \cap R = \emptyset$. As before, this implies $|T - R| \geq n - 1$, a contradiction.

Case 2: Suppose $y_3 \leq b_{n+k} \leq b_p$. Since C is a strict alternating cycle, we know $(x_1, y_2) \in \text{Inc}(A, B)$. Then for every $\beta \in [n - 1]$, we have $y_2 < b_{1-\beta} \leq b_{n+k}$. It follows from Lemma 6.5 that all $n - 1$ pairs in the following set belong to T :

$$N'' = \{(x_2, b_{1-\beta}) : \beta \in [n - 1]\}.$$

All pairs in N'' are adjacent to (x_1, y_1) , so $N'' \subseteq T - S$. Furthermore, at most one pair from N'' is in R as all pairs in $R - S$ end at b_{n+k} . This implies that $|T - R| \geq n - 2$, a contradiction.

With contradictions in all cases, this completes the proof when the first of the two statements applies. By reviewing the steps in the argument, the proof when the second of the two statements holds is symmetric. This completes the proof of the lemma. \square

The natural next step is to include an analogous result for expansions. This would require a separate proof as we are working only with sets in MSINR_{D_3} . However, such a result is not so straightforward. The following lemma for expansions shows that MSINR_{D_3} is closed under expansions with one possible exception. Although there is one new wrinkle in the argument, many elements are quite similar to the preceding proof and, in those situations, we will be brief.

Lemma 6.9 *Let $S \in \text{MSINR}_{D_3}$ and $i \in [n + k]$. Under the assumption that $|S| > (k + 1)(k + 2)/2 + 2 - n$, either $\text{DFEL}(i, S)$ and $\text{DLEF}(i, S)$ are both in MSINR or one of the following holds:*

- $\text{DFEL}(i, S)$ is reversible with $|\text{DFEL}(i, S)| \geq |S|$ and each strict alternating cycle $\{(x_\alpha, y_\alpha) : \alpha \in [3]\}$ in $\mathbb{C}_{D_3}(S)$ has an $\alpha \in [3]$ such that $x_\alpha = a_i$ and $y_{\alpha+2} = b_{i+k+1}$. Furthermore, each of these cycles C has a corresponding cycle C' also in S where $C' = (C - \{(x_{\alpha+1}, y_{\alpha+1})\}) \cup \{(x_{\alpha+1}, y)\}$ where the size of $(y, x_{\alpha+2})$ is 2.
- $\text{DLEF}(i, S)$ is reversible with $|\text{DLEF}(i, S)| \geq |S|$ and each strict alternating cycle $\{(x_\alpha, y_\alpha) : \alpha \in [3]\}$ in $\mathbb{C}_{D_3}(S)$ has an $\alpha \in [3]$ such that $x_\alpha = a_{i-1}$ and $y_{\alpha+2} = b_{k+i}$. Furthermore, each of these cycles C has a corresponding cycle C' also in S where $C' = (C - \{(x_{\alpha+1}, y_{\alpha+1})\}) \cup \{(x, y_{\alpha+1})\}$ where the size of (y_α, x) is 2.

Proof If $\text{DFEL}(i, S)$ or $\text{DLEF}(i, S)$ does not belong to MSINR , then one of the following two statements applies:

- (1) $\text{DFEL}(i, S)$ is reversible and $|\text{DFEL}(i, S)| \geq |S|$, or
- (2) $\text{DLEF}(i, S)$ is reversible and $|\text{DLEF}(i, S)| \geq |S|$.

We show that if the first of these two statements holds, then we either obtain a contradiction or we discover the structure of the alternating cycles described in the first statement of the lemma. The proof for the second statement is analogous.

After a relabeling, we may assume $i = 1$. If $\text{DFEL}(1, S)$ is reversible, then every strict alternating cycle $\{(x_\alpha, y_\alpha) : \alpha \in [3]\}$ in $\mathbb{C}_{D_3}(S)$ must have $x_\alpha = a_1$ for some $\alpha \in [3]$. Arbitrarily fix a cycle $C \in \mathbb{C}_{D_3}(S)$ and relabel its pairs so that $x_1 = a_1$.

Let R denote the reversible set $\text{DFEL}(1, S)$ where $(x_1, y_1) \in \text{FEBP}(1, S)$. Choose $T \in \text{MIR}$ with $R \subseteq T$. Then T contains the pairs (x_2, y_2) and (x_3, y_3) . Based on the size of S and the relationship $|S| \leq |R| \leq |T|$, we conclude T must be a canonical reversible set wherein $|T - R| < n - 2$. Let $m, p \in [n + k]$ be the integers specified by Lemma 6.5.

We pause to note that if $(x, y) \in R - S$, then $y = b_{k+2}$. As S was maximal, then there must also be some $(a, b) \in S - R$ which is adjacent to (x, y) . Clearly, this requires $(a, b) \in \text{FEBP}(1, S)$, so $a = a_1$.

We proceed by breaking the argument into cases based on the location of b_{m+k} .

Case 1: Suppose $y_3 \leq b_{n+k} \leq b_{m+k}$. It follows that all $n - 1$ pairs in the following set belong to T :

$$M = \{(x_2, b_{1-\beta}) : \beta \in [n - 1]\}.$$

All pairs in M are adjacent to (x_1, y_1) , so $M \subseteq T - S$. However, at most one pair from M belongs to $R - S$ since all pairs in $R - S$ end at b_{k+2} . This implies $|T - R| \geq n - 2$, a contradiction.

Case 2: Suppose $y_3 \leq b_{m+k} < b_{n+k}$. Let $y_1 = b_j$. We further subdivide this case based on the location of a_{p-k} .

First suppose $a_{j+1} < a_{p-k} \leq a_m$. Using the same definitions and notation from the proof of Lemma 6.8, it follows that $T - S$ contains all pairs in the following set:

$$N' = \{(a_m, b_{p+1-\beta} : \beta \in [s_1 + t_1]) \cup \{(a_{p-k-1+\gamma}, b_p) : \gamma \in [s_1 + t_3]\}.$$

Because $a_1 \leq a_j < a_{j+1} < a_{p-k}$, we conclude $b_p \neq b_{k+2}$. So at most 1 pair in N' belongs to R . Since $|N'| \geq n - 1$, this implies that $|T - R| \geq n - 2$, a contradiction.

Next consider the case where $a_{p-k} \leq a_{j+1} \leq x_2$. Observe that $T - S$ contains all $n - 1$ pairs in the following set:

$$M' = \{(a_{j+\gamma}, b_p) : \gamma \in [n - 1]\}.$$

Because $x_1 < y_3$ in S_n^k and $x_1 = a_1$, we have $b_{k+2} \leq y_3 \leq b_p < x_1$. If $b_p \neq b_{k+2}$, then $M' \not\subseteq R$ which implies $|T - R| \geq n - 1$, a contradiction.

Therefore, we may assume $y_3 = b_{k+2} = b_p$. Recall the original choice of C was arbitrary with pairs relabeled so that $x_1 = a_1$. If $\text{DFEL}(1, S)$ is not in MSINR , we explore the characteristics of cycles in S . Furthermore, if $x_3 = a_\ell$, then $(x_2, b_{\ell-1})$ is in T by Lemma 6.5. As $(x_2, b_{\ell-1})$ is not adjacent to any pair of the form (a_1, y) , we may conclude $(x_2, b_{\ell-1}) \in S$. Thus S contains the cycle $\{(a_1, y_1), (x_2, b_{\ell-1}), (a_\ell, b_{k+2})\}$. A similar argument holds if $\text{DFEL}(1, S)$ is not in MSINR . □

Fix a cycle $C = \{(x_\alpha, y_\alpha) : \alpha \in [3]\}$ in \mathbb{C}_{D_3} . For each $\alpha \in [3]$, the set $\{(x_\alpha, w) : x_\alpha \leq w \leq y_\alpha\}$ is called the *forward α -fan for C* . Analogously, the set $\{(z, y_\alpha) \in \text{Inc}(A, B) : x_\alpha \leq z \leq y_\alpha\}$ is the *backward α -fan for C* . In general, when $S \in \text{MSINR}_{D_3}(C)$ and $\alpha \in [3]$, there is no reason that S should contain either the forward α -fan or the backward α -fan for C . However, Lemma 6.10, which follows from Lemma 6.8, guarantees that MSINR_{D_3} contains sets with fans.

Lemma 6.10 *Let $C \in \mathbb{C}_{D_3}$, and let $f : [3] \rightarrow \{f, b\}$. Then there is a set $S \in \text{MSINR}_{D_3}(C)$ such that for each $\alpha \in [3]$, S contains the forward α -fan for C if $f(\alpha) = f$ and S contains the backward α -fan for C if $f(\alpha) = b$.*

Proof We simply start with any set $S \in \text{MSINR}_{D_3}(C)$ and repeatedly apply the contraction lemma, retaining the cycle C at each step. By Lemma 6.8, the result is another set in $\text{MSINR}_{D_3}(C)$. For example, say we desire a set with the forward 1-fan for C . Clearly $(x_1, y_1) \in S$. Of all points u_i on the circle with (1) $x_1 \leq u_i \leq y_1$ and (2) $(x_1, w) \in S$ for all w with $u_i \leq w \leq y_1$, choose u_m as the unique one for which the size of (x_1, u_m) is minimum.

If $x_1 = a_m$, then S contains the required forward 1-fan and we move on to other values of α . If instead $x_1 \neq a_m$, then $(x_1, b_m) \in \text{LCBP}(m, S)$. Furthermore, $S' = \text{DFCL}(m, S)$ contains C , all pairs (x_1, w) with $u_{m-1} \leq w \leq y_1$, and all pairs belonging to the other two fans. We can repeatedly apply these modifications until the desired set is obtained. □

With Lemmas 6.8, 6.9, and 6.10 in hand, we proceed to show $C^* \in \mathbb{C}_{D_3}$. For each $C \in \mathbb{C}_{D_3}$ with $C = \{(x_\alpha, y_\alpha) : \alpha \in [3]\}$, the pairs $\{(y_\alpha, x_{\alpha+1}) : \alpha \in [3]\}$ are considered the

“gaps” of C . Define $g_\alpha(C)$ to be the size of the gap $(y_\alpha, x_{\alpha+1})$ and define the spread of C to be the quantity

$$\max\{g_\alpha(C) - g_{\alpha+1}(C) - g_{\alpha+2}(C) : \alpha \in [3]\}.$$

In turn, we let maxspread be the maximum value of the spread of C taken over all $C \in \mathbb{C}_{D_3}$. Since $2 \leq g_\alpha(C) \leq n$ for all $C \in \mathbb{C}_{D_3}$ and all $\alpha \in [3]$, then maxspread is at most $n - 4$. Note that if $C^* \in \mathbb{C}_{D_3}$, then maxspread = $n - 4$, but to establish this will take some work. First we find a cycle $C \in \mathbb{C}_{D_3}$ with spread $n - 4$.

Claim 1 *There is some $C \in \mathbb{C}_{D_3}$ whose spread is $n - 4$. i.e. maxspread = $n - 4$.*

Proof Among all cycles in \mathbb{C}_{D_3} , let C be one with maximum spread. Fix $S \in \text{MSINR}_{D_3}(C)$. Toward a contradiction, suppose the spread of C is less than $n - 4$. Label the pairs in C so that the spread is $g_3(C) - g_1(C) - g_2(C)$.

If $g_3(C) < n$, then use Lemma 6.10 to obtain a set $S' \in \text{MSINR}_{D_3}(C)$ which contains the forward 3-fan for C and the backward 1-fan for C . If $(x_1, y_1) = (a_i, b_j)$ with $i \neq j$, then S' also contains the cycle $\{(a_{i+1}, y_1), (x_2, y_2), (x_3, y_3)\}$ which has larger spread than C , a contradiction. So we may assume $(x_1, y_1) = (a_i, b_i)$ and, similarly, we may assume $(x_3, y_3) = (a_r, b_r)$.

With $g_3(C) < n$, $(x_1, y_1) = (a_i, b_i)$, and $(x_3, y_3) = (a_r, b_r)$, it must be the case that $g_1(C) > 2$. Otherwise $x_2 < y_3$ in S_n^k , which is not possible because C is a strict alternating cycle. Likewise, we may conclude $g_2(C) > 2$.

Because C has maximum spread, we see $(a_i, b_{i+1}) \notin S$. This implies $(x_1, y_1) \in \text{LEBP}(i - k, S)$. As $g_3(C) < n \leq k$ and $(x_3, y_3) = (a_r, b_r)$, we conclude $x_3 \neq a_{i-k}$. Furthermore, $x_2 < y_1$, so $x_2 \neq a_{i-k}$. Therefore, $\text{DFEL}(i - k, S)$ contains the strict alternating cycle $\{(x_1, b_{i+1}), (x_2, y_2), (x_3, y_3)\}$ with spread larger than that of C . Therefore, $\text{DFEL}(i - k, S) \notin \text{MSINR}$. The only alternative, according to Lemma 6.9, is that each cycle in $\mathbb{C}_{D_3}(S)$ has a gap of size n . However, C has no such gap, a contradiction.

Lastly, suppose $g_3(C) = n$ and relabel the circle so $x_1 = a_1$ and $y_3 = b_{k+2}$. Because the spread of C is less than $n - 4$, either $g_1(C) > 2$ or $g_2(C) > 2$. First suppose $g_1(C) > 2$. (An analogous argument holds if $g_2(C) > 2$.) Because C has maximum spread, $(x_2, y_2) = (a_\ell, y_2) \in S$ but $(a_{\ell-1}, y_2) \notin S$. Therefore, $(x_2, y_2) \in \text{FEBP}(\ell, S)$. Now observe that $a_2 < a_\ell < a_{k+2}$ and $x_2 = a_\ell < y_1$ in S_n^k imply $b_{k+2} < b_{k+\ell} < y_1$. Thus $y_3 = b_{k+2} \neq b_{\ell+k}$ and $y_1 \neq b_{\ell+k}$. Therefore, $\text{DLEF}(\ell, S)$ contains the cycle $\{(x_1, y_1), (a_{\ell-1}, y_2), (x_3, y_3)\}$ which has spread larger than that of C . By the choice of C , this implies $\text{DLEF}(\ell, S)$ is not in MSINR . As $\text{DLEF}(\ell, S) \in \text{INR}$, the first bullet point of Lemma 6.9 must apply. So $g_1(C) = n$. Furthermore, there is another cycle C' in S with $g_3(C') = 2$ and $g_2(C') = g_2(C)$. Thus the spread of C is $n - g_1(C) - g_2(C)$ while the spread of C' is $n - 2 - g_2(C)$, which is larger as $g_1(C) > 2$. This contradicts the choice of C .

With a contradiction in every case, we may conclude that maxspread = $n - 4$. □

Let \mathbb{C}_{sp} denote the family of all $C \in \mathbb{C}_{D_3}$ which have spread $n - 4$. For each $C \in \mathbb{C}_{sp}$, we may assume that the pairs in C have been labeled so that $a_1 = x_1$ and $y_3 = b_{k+2}$. Let MSINR_{sp} consist of all $S \in \text{MSINR}_{D_3}$ for which there is some $C \in \mathbb{C}_{sp}$ with C contained in S .

Claim 2 *The strict alternating cycle $C^* = \{(a_1, b_1), (a_2, b_{k+1}), (a_{k+2}, b_{k+2})\}$ is in \mathbb{C}_{sp} .*

Proof Choose $C \in \mathbb{C}_{sp}$ for which the sum of the sizes of (x_1, y_1) and (x_3, y_3) is minimum. Note that $C = C^*$ if this sum is 2, so we assume the sum is larger than 2 and argue to a contradiction. We assume first that the size of (x_1, y_1) is at least 2. Let $S \in \text{MSINR}_{sp}(C)$ contain the forward 1-fan for C .

Let $y_1 = b_j$. Then $(x_1, b_{j-1}) \in S$. If $(a_j, y_2) \in S$, we get an immediate contradiction, since S contains the strict alternating cycle C' obtained from C by replacing (x_1, y_1) and (x_2, y_2) by (x_1, b_{j-1}) and (a_j, y_2) .

It follows that $(x_2, y_2) = (a_{j+1}, y_2) \in \text{FEBP}(j + 1, S)$. Because $j \geq 2$, we have $(x_3, y_3) \notin \text{LEBP}(j + 1, S)$. Also, $y_1 \neq b_{j+k+1}$ as $x_2 < y_1$ in S_n^k , so $S' = \text{DLEF}(j + 1, S)$ contains C' . Therefore, $\text{DLEF}(j + 1, S) \notin \text{MSINR}$, however $\text{DLEF}(j + 1, S) \in \text{INR}$. By Lemma 6.9, the only alternative is for $\text{DFEL}(j + 1, S)$ to be reversible. In this case, since $x_2 = a_{j+1}$, we must have $g_1(C) = n$, a contradiction. \square

Now that we have C^* in a set S from MSINR_{D3} , we can steer toward the hypotheses in Lemma 6.7, seeking a set $S \in \text{MSINR}_{D3}$ with $A(S) \subseteq \{a_1, a_2, \dots, a_{k+2}\}$ and $B(S) \subseteq \{b_1, b_2, \dots, b_{k+2}\}$. We establish one condition at a time.

Claim 3 *There is some $S \in \text{MSINR}(C^*)$ with $A(S) \subseteq \{a_1, a_2, \dots, a_{k+2}\}$.*

Proof Let S be any set in $\text{MSINR}(C^*)$ which contains the backward 2-fan for C^* . For a proof by contradiction, suppose $(a, b) \in S$ with $a_{k+2} < a < a_1$. Since S contains (a_2, b_{k+1}) and S is independent, the intervals (a, b) and (a_2, b_{k+1}) must overlap. Let $b = b_j$. Then $1 \leq j \leq k$. However, since S contains the backward 2-fan for C , we have $(a_{j+1}, b_{k+1}) \in S$. This is a contradiction since (a, b) and (a_{j+1}, b_{k+1}) are adjacent. \square

Claim 4 *There is some $S \in \text{MSINR}(C^*)$ with $A(S) \subseteq \{a_1, a_2, \dots, a_{k+2}\}$ and $B(S) \subseteq \{b_1, b_2, \dots, b_{k+2}\}$.*

Proof We consider only sets $S \in \text{MSINR}(C^*)$ for which $A(S) \subseteq \{a_1, a_2, \dots, a_{k+2}\}$, which exist by Claim 3. Among these sets, let S be one for which the number of pairs (a, b) with $b_{k+2} < b < b_1$ is minimum. We suppose this minimum number is positive and argue to a contradiction.

Of all pairs $(a, b) \in S$ with $b_{k+2} < b < b_1$, choose one of minimum size. Then there is some i with $3 \leq i \leq k + 1$ so that $a = a_i$ and $(a_i, b) \in \text{FCBP}(i, S)$.

We note that any pair $(x, y) \in \text{LCBP}(i, S)$ has $a_1 \leq x \leq a_{k+2}$. Furthermore, all pairs in $\text{DFCL}(i, S) - S$ end at b_{i-1} . Therefore, all pairs in $S' = \text{DFCL}(i, S)$ start in $\{a_1, \dots, a_{k+2}\}$, and there are fewer pairs in S' of the form (a, b) where $b_{k+2} < b < b_1$. As C^* is contained in S' , this contradicts the choice of S . \square

With Claim 4 in hand, we have shown that there is a set $S \in \text{MSINR}_{D3}$ with $A(S) \subseteq \{a_1, a_2, \dots, a_{k+2}\}$ and $B(S) \subseteq \{b_1, b_2, \dots, b_{k+2}\}$ that contains the strict alternating cycle C^* . This set S satisfies the hypotheses of Lemma 6.7. Therefore, $|S| \leq (k + 1)(k + 2)/2 + 2 - n$. However, throughout this subsection, we had assumed that each set in MSINR had size greater than $(k + 1)(k + 2)/2 + 2 - n$. The contradiction completes the proof of Theorem 1.6. Recall that this inequality is best possible as shown by the construction in Example 4.10.

7 Closing Remarks

At the outset of the paper, we remarked that we were able to prove Theorem 1.1 without settling the issue of whether $\alpha(G_n^k) = (k+1)(k+2)/2$. To do so, we first obtained the full characterization of sets in $\text{MINR}(n, k)$ when $k < n \leq 2k$. This resulted in the proof that such sets have size at most $2 + (2k+1-n)(2k+2-n)/2$, a quantity which is less than $(k+1)(k+2)/2$.

When $k \geq 2n-6$, it follows that $\dim(S_n^k) = 3$. So to show that $\dim(S_n^k) = \chi(G_n^k)$ in this range, it is only necessary to prove that G_n^k is not 2-colorable. However, this follows from a theorem of Cogis [2] which asserts that if P is a poset and G is the associated graph of critical pairs, then $\dim(P) \leq 2$ if and only if $\chi(G) \leq 2$. This is a non-trivial theorem and the notation and terminology used by Cogis are quite different from the style and contents of this paper. A more combinatorial proof of this result can be found in [5]. Regardless, when $k \geq 2n-6$, it is an elementary exercise to show directly that $\chi(G_n^k) \geq 3$ by identifying an odd cycle contained in this graph.

When $n \leq k < 2n-6$, we have $\dim(S_n^k) = 4$. We were able to show that $\chi(G_n^k) = 4$ in this range by showing that G_n^k has so many triangles that no independent set could possibly intersect them all. This construction is a much more substantive exercise.

As for open problems, there are two obvious challenges stemming from our work. We have not been able to completely determine the sets which belong to $\text{MINR}(n, k)$ when $n \leq k$, and we are not even certain that there is any reasonable way in which this can be done. Also, when $k < n \leq 2k$, it would be interesting to find a way to prove the inequality in Theorem 1.5 without first finding all sets in $\text{MINR}(n, k)$.

As noted in several recent research papers, dimension can be defined for a set of incomparable (or critical) pairs in a poset, with the dimension of the poset being the special case where we consider the entire set of incomparable pairs. Accordingly, it would be of interest to determine whether $\dim(S) = \chi(S)$ for subsets $S \subseteq \text{Inc}(A, B)$. We expect the answer to be negative.

Finally, we should mention the classic problem of determining whether there is some constant $c_0 \geq 3$ for which there is an infinite sequence $\{P_n : n \geq 3\}$ of posets such that for each $n \geq 3$, (1) $\dim(P_n) \geq n$; and (2) if G_n is the graph of critical pairs of P_n , then $\chi(G_n) \leq c_0$. If the answer is negative, then it will become an interesting challenge to investigate classes of posets for which the dimension can be bounded as a function of the chromatic number of the associated graph of critical pairs.

Acknowledgments The research reported here was initiated in two informal workshops hosted by the United States Military Academy (West Point) in 2015 and 2016, and we thank them for providing a stimulating and encouraging atmosphere.

Smith acknowledges support from NSF-DMS grant #1344199. Harris acknowledges support from NSF-DMS grant #1620202.

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