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MAXIMAL DIMENSIONAL PARTIALLY ORDERED SETS III: A CHARACTERIZATION OF HIRAGUCHI'S INEQUALITY FOR INTERVAL DIMENSION

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Dushnik and Miller defined the dimension of a partially ordered set X, denoted dim X, as the smallest positive integer t for which there exist t linear extensions of X whose intersection is the partial ordering on X. Hiraguchi proved that if $n \geqslant 2$ and $|X| \leqslant 2n + 1$, then dim $X \leqslant n$. Bogart, Trotter and Kimble have given a forbidden subposet characterization of Hiraguchi's inequality by determining for each $n \geqslant 2$, the minimum collection of posets $\mathcal{C}_n$ such that if $|X| \leqslant 2n + 1$, then dim $X < n$ unless X contains one of the posets from $\mathcal{C}_n$. Although $|\mathcal{C}_3| = 24$, for each $n \geqslant 4$, $\mathcal{C}_n$ contains only the crown S_n^0 – the poset consisting of all 1 element and $n - 1$ element subsets of an n element set ordered by inclusion. In this paper, we consider a variant of dimension, called interval dimension, and prove a forbidden subposet characterization of Hiraguchi's inequality for interval dimension: If $n \geqslant 2$ and $|X| \leqslant 2n + 1$, the interval dimension of X is less than n unless X contains S_n^0.

1. Introduction

Dushnik and Miller [3] defined the dimension of a partially ordered set (poset) X, denoted dim X, as the smallest positive integer t for which there exists t linear extensions $L_1, L_2, ..., L_t$ of X whose intersection is the partial ordering on X, i.e. $x < y$ in X iff $x < y$ in L_i for each $i \leqslant t$.

If $\mathcal{C}$ is a collection of closed intervals of the real line R (points are considered to be closed intervals), then there is a natural partial ordering on $\mathcal{C}$ defined by $A \rhd B$ iff $a \in A$ and $b \in B$ imply $a > b$ in R. We then define the interval dimension of a poset X, denoted Idim X, to be the smallest positive integer t for which there exists a function F which assigns to each $x \in X$ a sequence $F(x)(1), F(x)(2), ..., F(x)(t)$ of closed intervals of R so that $x > y$ in X iff $F(x)(i) \rhd F(y)(i)$ for each $i \leqslant t$. It follows easily that Idim X = Idim $\hat{X} \leqslant$ dim X = dim $\hat{X}$ where $\hat{X}$ denotes the dual of X.

A poset X for which Idim $X = 1$ is called an interval order. In this paper we denote an n element chain by **n** and the free sum of posets X and Y by $X + Y$. With this notation we have the following characterization theorem for interval orders.

Theorem 1.1 (Fishburn [4]). *A poset is an interval order iff it does not contain* **2** + **2**.

In this paper we will find it convenient to use the concept of the join of two posets X and Y, denoted $X \oplus Y$. As defined in [8], $X \oplus Y$ is the poset whose point set is the same as the free sum $X + Y$. However to the partial order on $X + Y$, we add the relation $x < y$ for every $x \in X$, $y \in Y$. (This poset is what Birkhoff calls the lexicographic sum of X and Y over the poset **2**.) It is easy to see that dim $X \oplus Y = \max\{\dim X, \dim Y\}$ and Idim $X \oplus Y = \max\{\text{Idim } X, \text{Idim } Y\}$.

2. Hiraguchi's theorem and forbidden subposets

In 1955, Hiraguchi [5] proved the following theorem.

Theorem 2.1. *If* $|X| \geqslant 4$, *then* dim $X \leqslant [\frac{1}{2}|X|]$.

A poset X is said to be irreducible if $\dim(X - x) < \dim X$ for every $x \in X$. There are no irreducible posets of dimension 2 on 4 or 5 points since a poset has dimension $\geqslant 2$ iff it contains **1** + **1**. There are 24 non-isomorphic irreducible posets of dimension 3 on 6 or 7 points [10]. However for $n \geqslant 4$, there are no irreducible posets of dimension n on $2n + 1$ points [6], and only one poset of dimension n and $2n$ points [2]. This poset consists of all one element and $n - 1$ element subsets of an n element set ordered by inclusion; following the notation introduced in [7], we label this poset S_n^0. For $n \geqslant 2$, S_n^0 has maximal elements $A = \{a_1, a_2, ..., a_n\}$ and minimal elements $B = \{b_1, b_2, ..., b_n\}$ with a_i covering b_j iff $i \neq j$. We then have:

Theorem 2.2. *If* $|X| \leqslant 5$, *then* dim $X < 2$ *unless X contains* **1** + **1**. *If* $|X| \leqslant 7$, *then* dim $X < 3$ *unless X contains one of the 24 posets cata-*

logued in [10]. *If* $n \geqslant 4$ *and* $|X| \leqslant 2n + 1$, *then* dim $X < n$ *unless* X *contains* S_n^0.

This "forbidden subposet" characterization of Hiraguchi's inequality is very difficult to prove. The most difficult aspect of the proof is to show that although there are 21 irreducible posets of dimension three on seven points, there are no irreducible posets of dimension four on nine points.

From the inequality given in Section 1 and the characterization theorem for interval orders, we conclude:

Theorem 2.3. *If* $|X| \geqslant 2$, *then* Idim $X \leqslant [\frac{1}{2}|X|]$.

The primary purpose of this paper is to prove a forbidden subposet characterization of this inequality which will avoid much of the pathology encountered in the proof of Theorem 2.3. We will also obtain new inequalities for interval dimension; for some of these we will also obtain forbidden subposet characterizations.

3. Some preliminary inequalities

We begin this section by starting a number of inequalities for ordinary and interval dimension. Proofs may be found in [1,2,5,6,9,13].

Theorem 3.1. *Let* X *be a poset,* $x \in X$, C *a chain in* X, A *an antichain, and* M *the set of maximal elements. Then the following inequalities hold:*

(1) dim $X \leqslant 1 + \dim(X - x)$,
(2) Idim $X \leqslant 1 + \text{Idim}(X - x)$,
(3) dim $X \leqslant 2 + \dim(X - C)$,
(4) Idim $X \leqslant 2 + \text{Idim}(X - C)$,
(5) dim $X \leqslant |X - A|$ *when* $|X - A| \geqslant 2$,
(6) dim $X \leqslant$ *width* X,
(7) dim $X \leqslant 2$ *width* $(X - A) + 1$ *when* $X - A \neq \emptyset$,
(8) Idim $X \leqslant 2$ *width* $(X - A) - 1$ *when* $X - A \neq \emptyset$,
(9) dim $X \leqslant$ *width* $(X - M) + 1$ *when* $X - M \neq \emptyset$,
(10) Idim $X \leqslant$ *width* $(X - M)$ *when* $X - M \neq \emptyset$.

We comment that all the inequalities of Theorem 3.1 are known to be best possible except statement (8).

Let x and y be distinct points of a poset X; we say that x and y have the same holdings in X if $z > x$ iff $z > y$ for every $z \in X - \{x, y\}$ and $z < x$ iff $z < y$ for every $z \in X - \{x, y\}$. The following statement is proved in [9].

Theorem 3.2. *If x and y have the same holdings in a poset X, then* $\dim(X - x) = \dim X$ *unless $x \, I \, y$ in X and $X - x$ is a chain. In this case* $\dim X = 2$ *and* $\dim(X - x) = 1$.

For interval dimension we have the following variant of Theorem 3.2.

Theorem 3.3. *If x and y have the same holdings in a poset X, and $x \, I \, y$, then* $\text{Idim } X = \text{Idim}(X - x)$.

Proof. Let F be an interval coordinatization of $X - y$ of length $t = \text{Idim}(X - x)$. Extend F to X by defining $F(x) = F(y)$.

We note that if x and y have the same holdings but $x > y$, then the removal of x (or y) may decrease the interval dimension of X by one. The poset $2 + 2$ is just one special case where this situation occurs.

The following inequality is proved in [5].

Theorem 3.4. *If a is maximal element, b is a minimal element, $a \, I \, b$, and $X - \{a, b\} \neq \emptyset$, then* $\dim X \leqslant 1 + \dim(X - \{a, b\})$.

A stronger version of Theorem 3.4 holds for interval dimension. An incomparable pair a, b is said to satisfy property M if $z > a$ implies $z > b$ and $y < b$ implies $y < a$. We then have [13]:

Theorem 3.5. *If a, b satisfies property M, then* $\text{Idim } X \leqslant 1 + \text{Idim}(X - \{a, b\})$.

If $X = Y + Z$, then $\dim X = \max\{\dim Y, \dim Z\}$ unless both X and Y are chains; in this case, $\dim X = 2$. For interval dimension the corresponding statement is $\text{Idim } X = \max\{\text{Idim } Y, \text{Idim } Z\}$ unless both Y and Z are interval orders and each contains 2; in this case $\text{Idim } X = 2$.

The following statement follows immediately from the characterization theorem for interval orders.

Lemma 3.6. *If A is an antichain of a poset X and $|X - A| \leq 1$, then* Idim $X = 1$.

The following theorem then follows easily from Lemma 3.6 and Theorem 3.1(2).

Theorem 3.7. *If A is an antichain of a poset X and $|X - A| = n \leq 1$, then* Idim $X \leq n$.

We now derive a generalization of Theorem 3.1(10). If A is an antichain of a poset X, we let $X_U(A) = \{x \in X\colon x > a$ for some $a \in A\}$ and $X_L(A) = \{x \in X\colon x < a$ for some $a \in A\}$. Note that if A is a maximal antichain, then $X = X_U(A) \cup A \cup X_L(A)$ is a partition of X.

Theorem 3.8. *If A is a maximal antichain of X and every point of $X_U(A)$ is greater than every point of $X_L(A)$, then* Idim $X = \max\{\text{Idim}(X - X_L(A)), \text{Idim}(X - X_U(A))\}$.

Proof. Let F be an interval coordinatization of length t of $X - X_L(A)$ and G an interval coordinatization of length s for $X - X_U(A)$. Without loss of generality, we assume $s \leq t$. If $s < t$, define $G(x)(i) = G(x)(s)$ for every $x \in X - X_U(A)$ and integer i with $s < i \leq t$.

Then for each $i \leq t$, let P_i be the partial order defined by $x > y$ in P_i iff $x, y \in X - X_L(A)$ and $F(x)(i) \triangleright F(y)(i)$, $x, y \in X - X_U(A)$ and $G(x)(i) \triangleright G(y)(i)$, or $x \in X_U(A)$ and $y \in X_L(A)$. It follows by Theorem 1.1 that the poset (X, P_i) is an interval order. We then define an interval coordinatization H of length t for X by choosing intervals $H(x)(i)$ so that for each $i \leq t$, the intervals $\{H(x)(i)\colon x \in X\}$ form an interval coordinatization of the interval order (X, P_i).

We note that there is no analog of Theorem 3.8 for ordinary dimension as the examples given in [11] demonstrate.

The following result is easily established by slight modifications in the proof of Theorem 3.1(10) and the preceding theorem.

Theorem 3.9. *Suppose A is a maximal antichain of X, the width of $X_U(A)$ is $n \geq 1$, $X_U(A) = C_1 \cup C_2 \cup \ldots \cup C_n$ is a partition into chains, and $X_L(A) \cup C_i$ is a chain for some $i \leq n$. Then* Idim $X \leq n$.

If $a > b$ and $a \geq x \geq b$ implies $x = a$ or $x = b$, then a is said to cover b and the pair a, b is called a cover. The rank of a cover is the number of pairs x, y where a covers x, y covers b, and $x \not| y$. Hiraguchi proved that the removal of a cover of rank zero or one reduces the dimension of a poset at most one. For interval dimension we have:

Theorem 3.10. *If a, b is a cover of rank zero, then* Idim $X \leq 1 +$ Idim$(X - \{a, b\})$.

Proof. Let F be an interval coordinatization of $X - \{a, b\}$ of length t. For each $i \leq t - 1$, choose intervals $F(a)(i)$ and $F(b)(i)$ so that $x > y$ in X implies $F(x)(i) \rhd F(y)(i)$. Let P denote the partial order on X. Then let Q_1 be the partial order on $X - \{a, b\}$ defined by $x > y$ in Q_1 iff $F(x)(t) \rhd F(y)(t)$. Note Q_1 is an extension of the restriction of P to $X - \{a, b\}$.

Let

$$X_1 = \{x \in X: x < b \text{ in } X\},$$

$$X_2 = \{x \in X: x \,I\, b \text{ and } x < a \text{ in } X\},$$

$$X_3 = \{x \in X: x \,I\, b \text{ and } x \,I\, a \text{ in } X\},$$

$$X_4 = \{x \in X: x > b \text{ and } x \,I\, a \text{ in } X\},$$

$$X_5 = \{x \in X: x > a \text{ in } X\}.$$

We then consider each of these sets and the union of any collection of them as subposets of the interval order $(X - \{a, b\}, Q_1)$.

Now choose intervals $F(x)(t)$ and $F(x)(t + 1)$ for each $x \in X$ so that the intervals $\{F(x)(t): x \in X\}$ form an interval coordinatization of the interval order

$$(X_1 \cup X_2 \cup X_3) \oplus \{b\} \oplus X_4 \oplus \{a\} \oplus X_5$$

and the intervals $\{I'(x)(t+1): x \in X\}$ form an interval coordinatization of

$$X_1 \oplus \{b\} \oplus X_2 \oplus \{a\} \oplus (X_3 \cup X_4 \cup X_5).$$

It is straightforward to verify that F is an interval coordinatization of X of length $t + 1$.

We note that the removal of a cover of rank one may reduce the interval dimension by two as the poset S_3^0 shows.

4. The characterization theorems

We begin this section with a lemma which will be essential in our forbidden subposet characterization of Theorem 3.7.

Lemma 4.1. *If A is an antichain of a poset X and $|X - A| = n \geq 3$, then* Idim $X < n$ *unless A is a maximal antichain, one of the sets $X_U(A)$ and $X_L(A)$ is empty, and the other is an antichain.*

Proof. If A is not a maximal antichain we conclude dim $X < n$ from Theorem 2.3. Now suppose that A is maximal and that $X_U(A) \neq \emptyset \neq X_L(A)$. If there exists a pair $x \in X_U(A)$, $y \in X_L(A)$ with $x\, I\, y$, then there exists a pair $x_0 \in X_U(A)$, $Y_0 \in X_L(A)$ where x_0, y satisfies property M. Then Idim $X \leq 1 +$ Idim$(X - \{x_0, y_0\}) \leq 1 + (n-2) = n-1$. We conclude that all points in $X_U(A)$ are greater than all points in $X_L(A)$ and by Theorems 3.7 and 3.8, we have Idim $X \leq n - 1$.

Without loss of generality, we now assume that either $X_U(A) = \emptyset$ or $X_L(A) = \emptyset$ and our conclusion follows from Theorem 3.1(10).

We invite the reader to compare the following theorem with analogous result for ordinary dimension given in [12]. (See also Theorem 3.1(5).)

Theorem 4.2. *If A is an antichain of a poset X and $|X - A| = n \geq 2$, then* Idim $X < n$ *unless X contains S_n^0.*

Proof. Theorem 1.1 implies that the result holds for $n = 2$ since $S_2^0 =$ 2 + 2. Now let A be an antichain of a poset X with $|X - A| = n \geq 3$ and suppose that X does not contain S_n^0. By Lemma 4.1 and our earlier remarks on duality and free sums, we may assume without loss of generality that A is the set of maximal elements, $B = \{b_1, b_2, ..., b_n\} = X - A$ is the set of minimal elements, and both A and B are maximal antichains. Since X does not contain S_n^0, we may also assume that there does not exist a maximal element which covers all minimal elements except b_n. For each $b \in B$, we denote $\{a \in A : a\, I\, b\}$ by $I(b)$. Also let P denote the partial order on X and for each $i \leq n-1$, let Q_i be the extension of P which is the partial order on X defined by adding to P the comparabilities $b > a$ and $b > b_n > b_i$ for every $b \in B - \{b_i, b_n\}$ and every $a \in A$. By Theorem 1.1, we conclude that each poset (X, Q_i) is an interval order; we may then choose intervals $F(x)(i)$ for every $x \in X$ and $i \leq n-1$ so that the intervals $\{F(x)(i): x \in X\}$ form a coordinatization of (X, Q_i). It is easy to verify that F is an interval coordinatization of X of length t.

Lemma 4.3. *If A is an antichain of a poset X and $|X - A| = 4$, then* Idim $X < 3$ *unless one of the following statements is true:*

(1) X_U or X_L contains a three element antichain.

(2) X_U and X_L are each two element antichains and each point in $X_U(A)$ is greater than exactly one point in $X_L(A)$.

Proof. Suppose that A is an antichain of a poset X with $|X - A| = 4$ and Idim $X \geq 3$. If A is not maximal or either $X_U(A)$ or $X_L(A)$ is empty, the result follows from Theorem 4.2. Now suppose that $|X_U(A)| = |X_L(A)| = 2$. Then we conclude from Theorem 3.8 that there exists $x \in X_U(A)$, $y \in X_L(A)$ with $x\, I\, y$. Now if $X_U(A) = \{x_1, x_2\}$ and $x_1 > x_2$, then x_2 is incomparable with both points of $X_L(A)$ for if $x_2 > y$ for some $y \in X_L(A)$, by Theorem 1.1 we may remove the other point of $X_L(A)$ to produce a poset with interval dimension one.

Now x_1, x_2 is a cover of rank zero so we conclude that $X_L(A) = \{y_1, y_2\}$ is a two element antichain. If $x_1\, I\, y_1$ and $x_1\, I\, y_2$, then X is the free sum of posets each of which have interval dimension at most two. Therefore we may assume $x_1 > y_1$; in this case x_2, y_1 satisfies property M so we conclude that $x_1\, I\, y_2$.

It follows that X has the same interval dimension as a subposet of the poset shown in Fig. 1.

In view of our remarks on holdings given in Section 3, we see that this poset has the same ordinary dimension as a subposet of the poset in Fig. 2(a). However the diagram shown in Fig. 2(b) proves that the poset in (a) has ordinary dimension two.

We comment here that the Hasse diagram of the poset in Fig. 2(a) is a "tree". We refer the reader to [14] for theorems concerning such posets.

By duality we can now conclude that both $X_U(A) = \{x_1, x_2\}$ and $X_L(A) = \{y_1, y_2\}$ are two element antichains. Since it cannot be true that all points of $X_U(A)$ are greater than all points of $X_L(A)$, we may assume that $x_1 \| y_1$. Hence it follows that $x_2 \| y_2$. If $x_1 \| y_2$, then $x_2 \| y_1$ and X would be the free sum of components each of which has interval dimension at most two. Hence $x_1 > y_2$; similarly $x_2 > y_1$.

Now suppose $X_U(A) = \{x_1, x_2, x_3\}$ and $X_L(A) = \{y_1\}$. There are 5 posets on three points; we show that if $X_U(A)$ is any one of the four which are not antichains, then Idim $X < 3$.

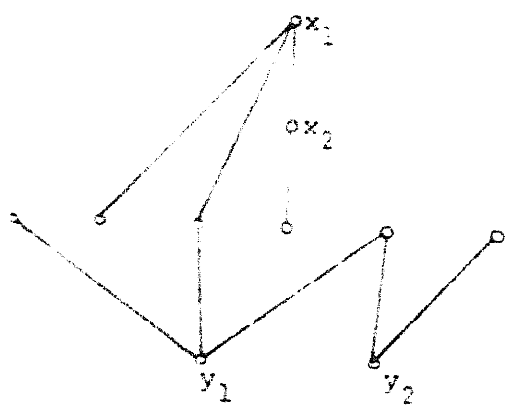

Fig. 1

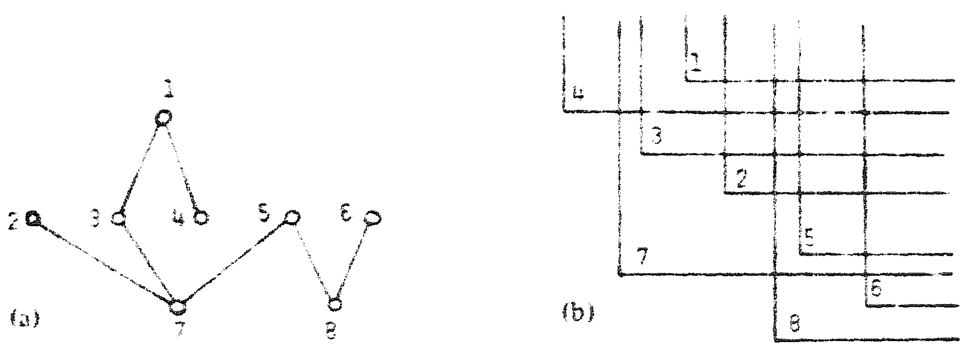

Fig. 2.

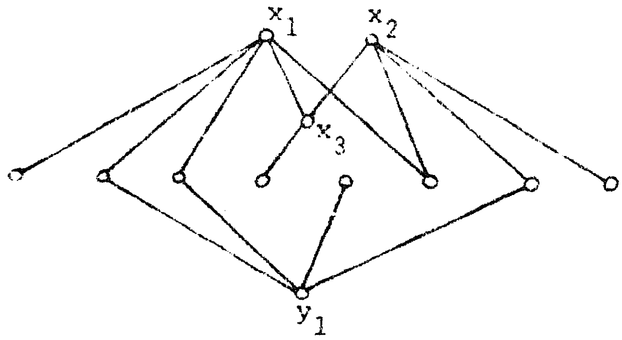

Fig. 3.

Suppose first that $X_U(A)$ is a chain. Then the removal of y_1 leaves a poset with interval dimension one. Now suppose the only order relation in $X_U(A)$ is $x_1 > x_2$. Then $\{x_1, x_2\}$ is a cover of rank zero which implies that $x_3 I y_1$. We then have that x_3, y_1 satisfies property M but $\mathrm{Idim}(X - \{x_3, y_1\}) = 1$.

Now suppose the only order relations in $X_U(A)$ are $x_1 > x_2$ and $x_1 > x_3$. It follows that $x_2 I y_1$, $x_3 I y_1$, but $x_1 > y_1$. Thus x_2, y_1 satisfies property M but $\mathrm{Idim}(X - \{x_2, y_1\}) = 1$.

Now suppose the only order relations in $X_U(A)$ are $x_1 > x_3$ and $x_2 > x_3$. Then it follows that $x_3 I y_1$ but $x_1 > y_1$ and $x_2 > y_1$. Thus X has the same interval dimension as a subposet of the poset in Fig. 3.

This poset contains three irreducible posets with ordinary dimension 3. However we conclude from Theorem 3.9 that it has interval dimension two.

Theorem 4.4. *If A is an antichain of a poset X and $|X - A| = n + 1 \geqslant 5$, then $\mathrm{Idim}\, X < n$ unless one of $X_U(A)$ and $X_L(A)$ contains an n element antichain.*

Proof. By Theorem 4.2, we may assume A is maximal. If either $X_U(A)$ or $X_L(A)$ is empty the result follows from Theorem 3.1(10). If all points of $X_U(A)$ are greater than all points of $X_L(A)$, the result follows from Theorem 3.8. So we assume there exists a pair x, y satisfying property M with $x \in X_U(A)$ and $y \in X_L(A)$. Then if both $X_U(A)$ and $X_L(A)$ contain at least two points, the result follows from Lemma 4.1.

Without loss of generality we may then assume that $X_U(A)$ contains n points and $X_L(A)$ only one. The conclusion that $X_U(A)$ is an n element antichain then follows easily by induction on n for if $x_1, x_2 \in X_U(A)$

and $x_1 > x_2$, we may remove $x_3 \in X_U(A) - \{x_1, x_2\}$ and decrease the interval dimension by at most one.

We are now ready to state and prove our forbidden subposet characterization of Hiraguchi's inequality for interval dimension. It is interesting to note that for the first time we will find it necessary to restrict the cardinality of an antichain.

Theorem 4.5. *If $n \geqslant 2$ and $|X| \leqslant 2n + 1$, then* Idim *$X < n$ unless X contains S_n^0.*

Proof. Our argument is by induction on n. Theorem 1.1 implies that the result holds for $n = 2$. Now assume validity if $n \leqslant k$ and suppose $n = k + 1 \geqslant 3$.

It is easy to see that we may assume without loss of generality that $|X| = 2n + 1$ and the width of X is n. Suppose first that there exists a maximum antichain $A = \{a_1, a_2, \ldots, a_n\}$ for which $X_U(A) \neq \emptyset \neq X_L(A)$. Then we may also assume that $X_U(A) = \{x_1, x_2, \ldots, x_n\}$ is a maximum antichain, $X_L(A) = \{y\}$, $y < a_n$, and $y \parallel a_1$. Since a_n, y is then a cover of rank zero, we label the remaining points so that $\{x_1, x_2, \ldots, x_{n-1}\} \cup \{a_1, a_2, \ldots, a_{n-1}\}$ form a copy of S_{n-1}^0 with x_i covering a_j iff $i \neq j$ for every $i, j \leqslant n - 1$.

Now a_1 is a minimal element, the pair x_1, a_1 satisfies property M, and therefore $X - \{x_1, a_1\}$ also contains S_{n-1}^0. Suppose $X - \{x_1, a_1, a_n\}$ is not S_{n-1}^0. Then $X - \{x_1, a_1, y\}$ is S_{n-1}^0 and we conclude that $a_n \parallel x_n$, $a_i < x_n$, $a_n < x_i$ for every i with $2 \leqslant i < n$. Then the pair x_2, a_2 satisfies property M and $X - \{x_2, a_2\}$ must also contain S_{n-1}^0. If $X - \{x_2, a_2, y\}$ is S_{n-1}^0, then we conclude $a_n < x_1$, $a_1 < x_n$ and thus $X - y$ is S_{n-1}^0. Therefore we may assume that $X - \{x_2, a_2, y\}$ is not S_{n-1}^0 in which case $X - \{x_2, a_2, a_n\}$ is S_{n-1}^0. This requires $a_1 < x_n$, $a_n \parallel x$, $y \parallel a_n$, and $y < a_i$ for every $i \leqslant n - 1$ with $i \neq 2$. But we have previously concluded that $a_n < x_2$ and hence $y < x_2$ also. This implies that $X - a_n$ is S_n^0. The contradiction shows that $X - \{x_1, a_1, a_n\}$ must be S_{n-1}^0.

Now we have that $y \parallel x_n$, $y < x_i$, and $a_i < x_n$ for every i with $2 \leqslant i < n$. Hence x_2, a_2 satisfies property M and $A - \{x_2, a_2\}$ must again contain S_{n-1}^0. If $X - \{x_2, a_2, a_n\}$ is S_{n-1}^0, then $X - a_n$ is S_n^0; if $X - \{x_2, a_2, a_n\}$ is not S_{n-1}^0, it follows easily that $X - y$ is S_n^0.

We may now assume that every maximum antichain consists entirely of minimal elements or entirely of maximal elements. Let x be the unique element of X which is neither minimal nor maximal. It follows that x must cover at least two minimal elements and be covered by at least two maximal elements. We label the maximal elements $A = \{a_1, a_2, ..., a_n\}$ and the minimal elements $B = \{b_1, b_2, ..., b_n\}$.

If all maximal elements are greater than all minimal elements, then $\mathrm{Idim}(X - x) = 1$. If $a \in A$, $b \in B$, and $a I b$, then $X - \{a, b\}$ must contain S_{n-1}^0 and it follows that $X - \{a, b, x\}$ is S_{n-1}^0 and $n \geq 5$. We may then assume that $a_n I b_n$ and $X - \{a_n, b_n, x\}$ is S_{n-1}^0 with $a_i I b_i$, and $a_i > b_j$ for every $i \neq j$ with $i, j \leq n - 1$. We also assume $a_1 > x, a_2 > x, x > b_3$, and $x > b_4$.

If $X - \{a_1, b_1, x\}$ and $X - \{a, b_2, x\}$ are both S_{n-1}^0, then $X - x$ is S_n^0. But if $X - \{a_1, b_1, a_2\}$ is S_{n-1}^0, so is $X - \{a_1, b_1, x\}$ and if $X - \{a_2, b_2, a_1\}$ is S_{n-1}^0, so is $X - \{a_2, b_2, x\}$. We conclude that $X - x$ is S_n^0 and the proof of our theorem is complete.

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