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Boolean Dimension, Components and Blocks

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Abstract
We investigate the behavior of Boolean dimension with respect to components and blocks. To put our results in context, we note that for Dushnik-Miller dimension, we have that if $\dim(C) \le d$ for every component C of a poset P, then $\dim(P) \le \max\{2, d\}$; also if $\dim(B) \le d$ for every block B of a poset P, then $\dim(P) \le d+2$. By way of constrast, local dimension is well behaved with respect to components, but not for blocks: if $\operatorname{ldim}(C) \le d$ for every component C of a poset P, then $\operatorname{ldim}(P) \le d+2$; however, for every $d \ge 4$, there exists a poset P with $\operatorname{ldim}(P) = d$ and $\dim(B) \le 3$ for every block B of P. In this paper we show that Boolean dimension behaves like Dushnik-Miller dimension with respect to both components and blocks: if $\operatorname{bdim}(C) \le d$ for every component C of P, then $\operatorname{bdim}(P) \le 2 + d + 4 \cdot 2^d$; also if $\operatorname{bdim}(B) \le d$ for every block of P, then $\operatorname{bdim}(P) \le 19 + d + 18 \cdot 2^d$.

Keywords Posets · Boolean dimension

1 Notation and Terminology

We consider combinatorial problems for finite posets. As has become standard in the literature, we use the terms *elements* and *points* interchangeably in referring to the members

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of the ground set of some poset $P = (X, <)$. We will write $x \parallel y$ when x and y are *incomparable* in P, and we let $\mathrm{Inc}(P)$ denote the set of all ordered pairs (x, y) with $x \parallel y$ in P. As a binary relation, $\mathrm{Inc}(P)$ is symmetric. The *dual* of a poset P will be denoted by P^*, while the dual of a linear order L on X by L^*. If L is a linear order on X and $Y \subseteq X$, then we will write $L(Y)$ for the restriction of L to Y. We will also use the notation $L = [A < B]$ when the elements of X can be labeled so that $L = [u_1 < u_2 < \cdots < u_m]$ and $A = [u_1 < u_2 < \cdots < u_k]$, $B = [u_{k+1} < u_{k+2} < \cdots < u_m]$ for some index k. This notation then generalizes naturally to an expression such as $L = [A_1 < A_2 < \cdots < A_s]$. For two elements $x, y \in P$ we say that x *covers* y if $y < x$ and there is no element $z \in P$ with $y < z < x$. The *cover* graph of P has vertex set the elements of P, and two vertices x and y are joined by an edge if one of them covers the other in P. Finally, we will also use the now standard notation $[n] = \{1, 2, \ldots, n\}$.

A nonempty family $\mathcal{R} = \{L_1, L_2, \ldots, L_d\}$ of linear extensions of P is called a *realizer* of P when $x \le y$ in P if and only if $x \le y$ in L_i for each $i = 1, 2, \ldots, d$. Clearly, $\mathcal{R}$ is a realizer if and only if for each $(x, y) \in \mathrm{Inc}(P)$, there is some i for which $x > y$ in L_i. The *dimension* of a poset P, as defined by Dushnik and Miller in their seminal paper [2], is the least positive integer d for which P has a realizer of size d. A subset S of $\mathrm{Inc}(P)$ is called *reversible* if there is a linear extension L of P with $x > y$ in L for every (x, y). When P is not a chain then the dimension of P is the least positive integer d for which there is a partition $\mathrm{Inc}(P) = S_1 \cup S_2 \cup \cdots \cup S_d$ with each S_i reversible. A subset $\{(x_\alpha, y_\alpha) : \alpha \in [k]\}$ of $\mathrm{Inc}(P)$ is called an *alternating cycle* if $x_\alpha \le y_{\alpha+1}$ for all $\alpha \in [k]$, with addition on the indicies understood cyclically. It is easy to see that alternating cycles are not reversible. On the other hand, Trotter and Moore [9] proved that they are the only obstructions for a set of incomparable pairs to be reversible, i.e., $S \subseteq \mathrm{Inc}(P)$ is reversible if and only if it does not contain an alternating cycle. For more details about now standard concepts and techniques for working with Dushnik-Miller dimension, the reader may consult any of several recent research papers, e.g., [4, 7] and [11] or the research monograph [8].

In recent years, researchers have been investigating combinatorial problems for two variations of Dushnik-Miller dimension, known as *Boolean dimension* and *local dimension*, respectively.

Let P be a poset with at least two elements and let $\mathcal{B} = \{L_1, L_2, \ldots, L_d\}$ be a nonempty family of linear orders (need not to be linear extensions of P) on the ground set of P. Also, let τ be a Boolean function which maps all 0-1 strings of length d to $\{0, 1\}$. For each ordered pair (x, y) of distinct elements of P, we form the bit string $q(x, y, \mathcal{B})$ of length d which has value 1 in coordinate i if and only if $x < y$ in L_i. We call the pair $(\mathcal{B}, \tau)$ a *Boolean realizer* of P if for every ordered pair (x, y) of distinct elements of P, we have $x < y$ in P if and only if $\tau(q(x, y, \mathcal{B})) = 1$. Nešetřil and Pudlák [5] (by slightly modifying the definition of Gambosi, Nešetřil and Talamo [3]) defined the *Boolean dimension of* P, denoted $\mathrm{bdim}(P)$, as the least positive integer d for which P has a Boolean realizer $(\mathcal{B}, \tau)$ with $|\mathcal{B}| = d$. By convention, the Boolean dimension of a one element poset is 1.

Clearly, $\mathrm{bdim}(P) \le \dim(P)$, since if $\mathcal{R} = \{L_1, L_2, \ldots, L_d\}$ is a realizer of P, we can simply take τ to be the function which maps the all ones bit string $(1, \ldots, 1)$ to itself while all other bit strings of length d are mapped to 0. Trivially, $\mathrm{bdim}(P) = 1$ if and only if P is either a chain or an antichain; if Q is a subposet of P, then $\mathrm{bdim}(Q) \le \mathrm{bdim}(P)$; and $\mathrm{bdim}(P) = \mathrm{bdim}(P^*)$. It is an easy exercise to show that if $\mathrm{bdim}(P) = 2$, then $\dim(P) = 2$, while Trotter and Walczak [10] proved the modestly more challenging fact that if $\mathrm{bdim}(P) = 3$, then $\dim(P) = 3$.

Again, let P be a poset. A *partial linear extension*, abbreviated *ple*, of P is a linear extension of a subposet of P. Whenever $\mathcal{L}$ is a family of ple's of P and $u \in P$, we set

$\mu(u, \mathcal{L}) = |\{L \in \mathcal{L} : u \in L\}|$. In turn, we set $\mu(P, \mathcal{L}) = \max\{\mu(u, \mathcal{L}) : u \in P\}$. A family $\mathcal{L}$ of ple's of a poset P is called a *local realizer* of P if for every pair (x, y) of distinct elements of the ground set of P, we have $x < y$ in P unless there is some $L \in \mathcal{L}$ with $x > y$ in L. The *local dimension of* P, denoted $\text{ldim}(P)$, is then defined[1] to be the least positive integer d for which P has a local realizer $\mathcal{L}$ with $\mu(P, \mathcal{L}) = d$.

Clearly, $\text{ldim}(P) \leq \dim(P)$ since every realizer P is also a local realizer. It is again easily seen that $\text{ldim}(P) = 1$ if and only if P is a chain; if Q is a subposet of P, then $\text{ldim}(Q) \leq \text{ldim}(P)$; and $\text{ldim}(P^*) = \text{ldim}(P)$. It is an easy exercise to show that if $\text{ldim}(P) = 2$, then $\dim(P) = 2$.

Recall that for $n \geq 2$, the *standard example* S_n is a height 2 poset with minimal elements $A = \{a_1, a_2, \ldots, a_n\}$, maximal elements $B = \{b_1, b_2, \ldots, b_n\}$ and $a_i < b_j$ in S_n if and only if $i \neq j$. As is well known, $\dim(S_n) = n$. On the other hand, it is another easy exercise to show that $\text{bdim}(S_n) = 4$ for all $n \geq 4$, while $\text{ldim}(S_n) = 3$ for all $n \geq 3$.

2 Statements of Results

To state our main results, we will need some basic concepts of graph theory, including connected and disconnected graphs, components, cut vertices, and k-connected graphs for an integer $k \geq 2$. Recall that when G is a graph, a connected induced subgraph H of G is called a *block* of G when H is maximal connected subgraph without a cut vertex, or quivalently it is either a maximal 2-connected subgraph, a bridge (with its ends) or an isolated vertex. Now a poset P is said to be *connected* if its cover graph is connected. A subposet B of P is said to be *convex* if $x, z \in B$ and $x < y < z$ in P imply $y \in B$. When B is a convex subposet of P, the cover graph of B is an induced subgraph of the cover graph of P. A convex subposet B of P is called a *component* of P when the cover graph of B is a component of the cover graph of P. A convex subposet B of P is called a *block* of P when the cover graph of B is a block of the cover graph of P. A point x in a poset P is called a *cut vertex* of P when x is a cut vertex of the cover graph of P.

As is well known, when P is a disconnected poset with components $C_1, C_2, \ldots, C_t$, then

$$\dim(P) = \max\{2, \max\{\dim(C_i) : 1 \leq i \leq t\}\}.$$

For local dimension, it is an easy exercise to show that

$$\text{ldim}(P) \leq 2 + \max\{\text{ldim}(C_i) : 1 \leq i \leq t\},$$

but we do not know whether this inequality is best possible.

We prove a corresponding, but somewhat more complicated, result for Boolean dimension. This is the first of our two main results.

Theorem 1 *Let P be a disconnected poset with components $C_1, C_2, \ldots, C_t$. If $d = \max\{\text{bdim}(C_i) : 1 \leq i \leq t\}$, then $\text{bdim}(P) \leq 2 + d + 4 \cdot 2^d$.*

The situation with blocks is more complex. For Dushnik-Miller dimension, Trotter, Walczak and Wang [11] proved that when P is a connected poset with blocks $B_1, B_2, \ldots, B_t$,

[1]The concept of local dimension is due to Torsten Ueckerdt [12] and was shared with participants of the workshop *Order and Geometry* held in Gułtowy, Poland, September 14–17, 2016. Ueckerdt's new concept resonated with participants at the workshop and served to rekindle interest in the notion of Boolean dimension as well.

then

$$\dim(P) \leq 2 + \max\{\dim(B_i) : 1 \leq i \leq t\}.$$

Furthermore, this inequality is best possible. Neither the proof of the inequality, nor the proof that the inequality is best possible is elementary. Surprisingly, however, there is no parallel result for local dimension, as Bosek, Grytczuk and Trotter [1] proved that for every $d \geq 4$, there is a poset P with $\text{ldim}(P) \geq d$, such that $\text{ldim}(B) \leq 3$ whenever B is a block in P.

The second of our two main results is the following theorem showing that Boolean dimension behaves like Dushik-Miller dimension and not like local dimension when it comes to blocks, i.e., we show that the Boolean dimension of a poset is bounded in terms of the maximum Boolean dimension among its blocks.

Theorem 2 *Let P be a poset with blocks $B_1, B_2, \ldots, B_t$. If $d = \max\{\text{bdim}(B_i) : 1 \leq i \leq t\}$, then* $\text{bdim}(P) \leq 19 + d + 18 \cdot 2^d$.

We doubt that the inequalities in Theorem 1 and Theorem 2 are sharp, but in some sense they are not so far from the truth, as we will show that for large d, there is a disconnected poset P with $\text{bdim}(P) = 2^{\Omega(d)}$ and $\text{bdim}(C) \leq d$ for every component C of P (and hence $\text{bdim}(B) \leq d$ for every block B of P).

3 Proofs

In discussing Boolean realizers for a poset P, the phrase "a pair (x, y)" will always refer to an ordered pair of distinct elements of P. Trivially, a poset P has Boolean dimension at most d, if P has a Boolean realizer $(\mathcal{B}, \tau)$ with $|\mathcal{B}| = d$. In defining a Boolean realizer, most of the work will go into the construction of the linear orders in the family $\mathcal{B}$. Typically, $\mathcal{B}$ will be made up of subfamilies of linear orders, each subfamily serving to reveal certain details concerning a pair (x, y). As far as the Boolean formula τ is considered, rather than explicitly writing out the rule for the function τ, we will simply explain how we can determine whether x is less than y in P based on the bits associated with the linear orders in $\mathcal{B}$. As we shall see, there are times when we know whether $x < y$ in P after seeing just a few of the bits in $q(x, y, \mathcal{B})$, while in other instances, we may need to see all or nearly all of the bits.

We will make frequent use of two simple lemmas. Both are standard tools in the field, but we nevertheless include the short proofs, as they are instructive for the more complex results to follow.

Lemma 1 *Let P be a poset with ground set X, and let ϕ be a t-coloring of X. Then there is a family $\mathcal{F} = \{N_1, N_2\}$ of two linear orders on X so that given a pair (x, y) of distinct elements of X, we can determine whether $\phi(x)$ is the same as $\phi(y)$ from the bits associated with the linear orders in $\mathcal{F}$.*

Proof We may, without loss of generality, assume ϕ uses the integers in $[t]$ as colors. For each $i \in [t]$, let X_i consist of all $x \in X$ with $\phi(x) = i$, and let L_i be an arbitrary linear order on X_i. Then set

$$N_1 = [L_1 < L_2 < L_3 < \cdots < L_{t-1} < L_t] \quad \text{and}$$
$$N_2 = [L_1^* < L_2^* < L_3^* < \cdots < L_{t-1}^* < L_t^*].$$

Now for the family $\mathcal{F} = \{N_1, N_2\}$, if $\phi(x) = \phi(y)$, we will get either (0, 1) or (1, 0), but if $\phi(x) \neq \phi(y)$, we will get either (0, 0) or (1, 1). □

Lemma 2 *Let P be a poset with ground set X, and let ϕ be a t-coloring of X. Then there is a family $\mathcal{F}$ of $4\lceil \log_2 t \rceil$ linear orders on X so that given a pair (x, y) of distinct elements of X, we can determine the pair $(\phi(x), \phi(y))$ of colors from the bits associated with the linear orders in $\mathcal{F}$.*

Proof Setting $r = \lceil \log_2 t \rceil$, we may, without loss of generality, assume ϕ uses the subsets of $[r]$ as colors. For each $j \in [r]$, let X_j consist of all $u \in X$ with $j \in \phi(u)$ and let L_0 be an arbitrary linear order on X. Then for each $j \in [r]$, we add the following four linear orders to the family $\mathcal{F}$:

$$\begin{aligned} M_1(j) &= [L_0(X_j) < L_0(X \setminus X_j)] \\ M_2(j) &= [L_0^*(X_j) < L_0(X \setminus X_j)] \\ M_3(j) &= [L_0(X \setminus X_j) < L_0(X_j)] \\ M_4(j) &= [L_0(X \setminus X_j) < L_0^*(X_j)] \end{aligned}$$

If $j \in \phi(x)$ and $j \in \phi(y)$, then the bits for the query will either be (1, 0, 1, 0) or (0, 1, 0, 1). If $j \in \phi(x)$ and $j \notin \phi(y)$, then the bits will be (1, 1, 0, 0). Conversely, if $j \notin \phi(x)$ and $j \in \phi(y)$, then we will have (0, 0, 1, 1). Finally, if $j \notin \phi(x)$ and $j \notin \phi(y)$, then the query will return either (1, 1, 1, 1) or (0, 0, 0, 0). Hence the $4r$ linear orders together will clearly enable us to determine the pair $(\phi(x), \phi(y))$. □

3.1 Boolean Dimension and Components

In this subsection, we prove Theorem 1.

Proof Let P be a disconnected poset with components $C_1, C_2, \ldots, C_t$, and assume that $\mathrm{bdim}(C_i) \leq d$, for each $i \in [t]$. Let X be the ground set of P, and for each $i \in [t]$ let X_i be the ground set of the component C_i. Furthermore, for each $i \in [t]$, let $(\mathcal{B}_i, \tau_i)$ be a Boolean realizer of C_i with $|\mathcal{B}_i| = d$. We label the linear orders in $\mathcal{B}_i$ as $\{L^i_j : j \in [d]\}$.

We now show that P has a Boolean realizer $(\mathcal{B}, \tau)$, with $|\mathcal{B}| = 2 + d + 4 \cdot 2^d$. The family $\mathcal{B}$ will be the union

$$\mathcal{B} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3,$$

where

$$|\mathcal{F}_1| = 2, \qquad |\mathcal{F}_2| = 4 \cdot 2^d, \qquad |\mathcal{F}_3| = d.$$

We begin by defining a coloring $\phi_1 : X \to [t]$ by setting $\phi_1(x) = i$ when $x \in X_i$. We use Lemma 1 to determine a family $\mathcal{F}_1$ of size 2 such that for each pair (x, y), the bits for the linear orders in $\mathcal{F}_1$ determine whether $\phi_1(x)$ is equal to $\phi_1(y)$.

Next consider the set $\mathbb{T} = \{\tau_i : i \in [t]\}$. Although the integer t is not bounded in terms of d, the size of $\mathbb{T}$ is at most 2^{2^d}. Therefore, the coloring $\phi_2 : X \to \mathbb{T}$ defined by setting $\phi_2(x) = \tau_i$ when $x \in X_i$ uses at most 2^{2^d} colors. Using Lemma 2, we take $\mathcal{F}_2$ as a family of $4 \cdot 2^d$ linear orders on X so that given a pair (x, y), we can determine the pair $(\phi_2(x), \phi_2(y))$ from the bits associated with the linear orders in $\mathcal{F}_2$.

Finally for each $j \in [d]$, let L_j be a linear order on X such that for each $i \in [t]$, the restriction of L_j to X_i is L^i_j, and take $\mathcal{F}_3 = \{L_j : j \in [d]\}$.

Now let (x, y) be a pair. From the bits associated with the linear orders in $\mathcal{F}_1$, we know whether x and y are in the same component or not. If not, then we know that x and y are incomparable in P. So we can assume that we have learned that x and y are in the same component. Next, from the bits associated with the linear orders in $\mathcal{F}_2$, we can learn the common color $\phi_2(x) = \phi_2(y)$, which is the truth function τ_i for the component C_i containing both x and y. Then we can apply the truth function τ_i to the bits for the linear orders in $\mathcal{F}_3$. Since the restriction of these linear orders to X_i is $\mathcal{B}_i$, this will finally answer whether x is less than y in C_i, and hence in P. This finishes the proof of Theorem 1. □

Now we explain why the bound in Theorem 1 cannot be improved dramatically. Consider a large integer n and the family $\mathbb{P}_n$ of all posets P of height at most 2 on the ground set $X = A \cup B$, with all elements of $A = \{a_1, a_2, \ldots, a_n\}$ being minimal in P and all elements of $B = \{b_1, b_2, \ldots, b_n\}$ being maximal in P. Clearly, there are 2^{n^2} such posets, since for each pair $(a_i, b_j) \in A \times B$, we can choose whether or not $a_i < b_j$ in P.

In [5], Nešetřil and Pudlák show that if P is a poset on $2n$ points, then $\text{bdim}(P) \leq c \log_2 n$ for some universal constant c, and they basically use the family $\mathbb{P}_n$ to show that this inequality is essentially best possible. This follows from the fact that there are not more then $((2n)!)^s \, 2^{2^s}$ Boolean realizers with s linear orders, and hence if $\text{bdim}(P) \leq s$ for all $P \in \mathbb{P}_n$, then we must have $((2n)!)^s \, 2^{2^s} \geq |\mathbb{P}_n| = 2^{n^2}$. However, this implies that $s = \Omega(\log_2 n)$.

Now consider the disconnected poset P formed by taking the disjoint sum of a copy of each poset in $\mathbb{P}_n$. Setting $d = c \log_2 n$, we then have $\text{bdim}(C) \leq d$ for every component C of P. On the other hand, we claim that $\text{bdim}(P) = 2^{\Omega(d)}$. To see this, suppose that $\text{bdim}(P) = m$ and let $(\mathcal{B}, \tau)$ be a Boolean realizer for P with $|\mathcal{B}| = m$. Now let Q be any poset from $\mathbb{P}_n$, and let $\mathcal{B}_Q$ be the family of linear orders obtained by taking the restrictions of the linear orders in $\mathcal{B}$ to the ground set of Q. Then $(\mathcal{B}_Q, \tau)$ is a Boolean realizer for Q. Since τ is now fixed, the number of realizers we can produce in such a way is at most $((2n)!)^m$. However, then we must have $(2n!)^m \geq |\mathbb{P}_n| = 2^{n^2}$, which implies $m = \Omega\left(\frac{n}{\log_2 n}\right) = 2^{\Omega(d)}$.

3.2 Boolean Dimension and Blocks

In this subsection, we prove Theorem 2. To start with, we first describe one of the key ideas, extracted from [11]. In the argument to follow, we will encounter the following situation. We will have a poset P with a cut vertex w and two connected convex subposets Q and Q', such that their ground sets, Y and Y', respectively, share only the element w. Then clearly the subposet Q'' of P with ground set $Y'' = Y \cup Y'$ is connected and convex, and the point w is a cut vertex of Q''. Then if $L = [A < w < B]$ and $L' = [C < w < D]$ are linear orders of Y and Y', respectively, there are many ways to determine a linear order L'' on Y'' such that $L''(Y) = L$ and $L''(Y') = L'$. However, in our argument, we will *always* do it using the following *merge rule*: $L'' = [A < C < w < D < B]$. It is important to note that this choice forces points in $A \cup B$ to the "outside" while concentrating points of $C \cup D$ in the "inside".

Now on to the proof. First let P be a connected poset with $\text{bdim}(B) \leq d$ for every block B of P. We will build a Boolean realizer $(\mathcal{B}, \tau)$ for P with $|\mathcal{B}| \leq 17 + d + 18 \cdot 2^d$. The family $\mathcal{B}$ will be the union

$$\mathcal{B} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_{11}$$

where:

$$|\mathcal{F}_1| = |\mathcal{F}_5| = |\mathcal{F}_6| = 2, \qquad |\mathcal{F}_2| = |\mathcal{F}_8| = |\mathcal{F}_9| = 4 \cdot 2^d, \qquad |\mathcal{F}_3| = d$$
$$|\mathcal{F}_4| = 3 \qquad |\mathcal{F}_7| = |\mathcal{F}_{11}| = 4,$$

and $\mathcal{F}_{10}$ is the union of $2 \cdot 2^d$ families, each of size 3. As a consequence, we will have $|\mathcal{B}| = 17 + d + 18 \cdot 2^d$, as required.

Let $\mathbb{B}$ denote the set of all blocks of P, and let $t = |\mathbb{B}|$. We may assume $t \geq 2$, otherwise P itself is a block and $\text{bdim}(P) \leq d$. Let $\mathbb{B} = \{B_1, B_2, \ldots, B_t\}$ be a labelling of the blocks in P so that whenever $2 \leq i \leq t$, block B_i has a (necessarily unique) point in common with $B_1 \cup B_2 \cup \cdots \cup B_{i-1}$. This point will be called the *root* of B_i and denoted $\rho(B_i)$. The block B_1 does not have a root. For each $i \in [t]$, we put X_i for the ground set of B_i, and we let $Y_i = X_1 \cup X_2 \cup \cdots \cup X_i$. We also set $Z_1 = X_1$ and $Z_i = X_i - \{\rho(B_i)\}$ for $2 \leq i \leq t$. Then clearly $Z_1 \cup Z_2 \cup \cdots \cup Z_t$ is a partition of X.

The first part of the proof will closely parallel the argument for Theorem 1. As before, we first define a coloring $\phi_1 : X \to [t]$ by setting $\phi_1(u) = i$ when $u \in Z_i$. Using Lemma 1, we then take $\mathcal{F}_1$ as a family of two linear orders so that given a pair (x, y), the bits for the linear orders in $\mathcal{F}_1$ determine whether $\phi_1(x) = \phi_1(y)$.

For each $i \in [t]$, let $(\mathcal{B}_i, \tau_i)$ be a Boolean realizer for B_i with $|\mathcal{B}_i| = d$. Then we again take the set $\mathbb{T} = \{\tau_i : i \in [t]\}$ which has size at most 2^{2^d}, and consider the coloring $\phi_2 : X \to \mathbb{T}$ defined by setting $\phi_2(x) = \tau_i$ when $x \in Z_i$. Just as before, this is a coloring using at most 2^{2^d} colors, so using Lemma 2, we can take $\mathcal{F}_2$ to be the family of $4 \cdot 2^d$ linear orders on X so that given a pair (x, y), we can determine the pair $(\phi_2(x), \phi_2(y))$ from the bits associated with the linear orders in $\mathcal{F}_2$.

Now we label the linear orders in $\mathcal{B}_i$ as $\{L^i_j : j \in [d]\}$. Recall that in the proof of Theorem 1, we chose a family $\{L_j : j \in [d]\}$ of linear orders on X such that for each $(i, j) \in [t] \times [d]$, the restriction of L_j to X_i is L^i_j. Here we must be more careful in the construction of these linear orders. For each $j \in [d]$, we define a linear order L_j on X using the following recursive procedure. First, set $L_j(1) = L^1_j$. Then suppose that for some $k \in [t-1]$, we have already defined a linear order $L_j(k)$ on the set Y_k. Now let $w = \rho(B_{k+1})$. Then w is both in Y_k and X_k, so there is a suitable A and B such that $L_j(k) = [A < w < B]$, and a suitable C and D such that $L^{k+1}_j = [C < w < D]$. We now can define $L_j(k+1)$ by the merge rule discussed previously, i.e., we put $L_j(k+1) = [A < C < w < D < B]$. At the end, when this procedure stops, we take $L_j = L_j(t)$ and set $\mathcal{F}_3 = \{L_j : j \in [d]\}$. Note that for each pair $(i, j) \in [t] \times [d]$, the restriction of L_j to X_i is still L^i_j.

We summarize what we have accomplished with the families $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$. Let (x, y) be a pair. If there is some $i \in [t]$ such that $x, y \in Z_i \subseteq X_i$, then this fact will be detected by the linear orders in $\mathcal{F}_1$. The linear orders in $\mathcal{F}_2$ will then detect the truth-function $\tau_i = \phi_2(x) = \phi_2(y)$ of B_i. Then, as the restriction of the linear orders in $\mathcal{F}_3$ is $\mathcal{B}_i$, we can determine whether x is less than y in P simply by applying τ_i to the bits for the linear orders in $\mathcal{F}_3$. As a consequence, for the balance of the argument, from now on we restrict our attention to pairs (x, y) satisfying the following property.

Property 1 $\phi_1(x) \neq \phi_1(y)$, i.e., there is no $i \in [t]$ for which $x, y \in Z_i$.

Next we define a digraph, which we will call the *root digraph of P*. Its vertex set is X and for each $2 \leq i \leq t$, we have an edge between $\rho(B_i)$ and every $u \in Z_i$ if u is comparable with $\rho(B_i)$ in P. The edge is directed from u to $\rho(B_i)$ when $u < \rho(B_i)$ in P and it is directed from $\rho(B_i)$ to u when $\rho(u_i) < u$ in P. Evidently, the root digraph of P is a directed forest.

The root digraph determines a poset Q whose ground set is X and u is covered by v in Q when there is an edge from u to v in the root digraph. Evidently, the poset Q is a “forest”, i.e., there are no cycles in the cover graph of Q. A well known theorem of Trotter and

Moore [9] asserts that the dimension of a poset whose cover graph is a forest is at most 3, so we add to $\mathcal{B}$ a family $\mathcal{F}_4$ which is a realizer of size 3 for Q.

Clearly, Q is a suborder of P, i.e., if $x < y$ (resp. $x > y$) in Q, as detected by the bits for $\mathcal{F}_4$ being (1, 1, 1) (resp. (0, 0, 0)), then $x < y$ (resp. $x > y$) in P, so for the balance of the argument, we restrict our attention to pairs (x, y) which also satisfy the following. property

Property 2 $x \parallel y$ in Q, i.e., the bits for the linear orders in $\mathcal{F}_4$ are not (1, 1, 1) or (0, 0, 0).

Now let (x, y) be a pair satisfying Property 1 and 2. We first decide on the relative position of x and y in the cover graph G of P. For this observe that there is a natural tree structure T on the Z_i's, with Z_i being a neighbour of Z_j when $\rho(B_i) \in Z_j$ or $\rho(B_j) \in Z_i$. We consider T as rooted at Z_1 and also fix a planar upward drawing of T. Then for every vertex of T there is a natural left-to-right ordering of its children. Let $i_x \neq i_y$ be such that $x \in Z_{i_x}$ and $y \in Z_{i_y}$. We can distinguish four cases:

- "x is below y" if Z_{i_x} is on the path from Z_{i_y} to Z_1 in T,
- "y is below x" if Z_{i_y} is on the path from Z_{i_x} to Z_1 in T,
- "x is left of y" if there is some Z_i which has two children Z_j and Z_k, with Z_j left of Z_k, such that Z_{i_x} is in the subtree rooted at Z_j and Z_{i_y} is in the subtree rooted at Z_k, and
- "y is left of x" which is defined analogously.

To identify which case are we in, let $[Z_{i_1}, Z_{i_2}, \ldots, Z_{i_t}]$ be the left-to-right and $[Z_{j_1}, Z_{j_2}, \ldots, Z_{j_t}]$ the right-to-left depth-first search order of the Z_i's according to T. For evety $i \in [t]$ choose an arbitrary linear order M_i on the elements of Z_i and put

$$\mathcal{F}_5 = \left\{[M_{i_1} < M_{i_2} < \cdots < M_{i_t}], [M_{j_1} < M_{j_2} < \cdots < M_{j_t}]\right\}.$$

Then, looking at the bits associated to the linear orders in $\mathcal{F}_5$, we see (1, 1) exactly if x is below y, (0, 0) exactly if y is below x, (1, 0) exactly if x is left of y and (0, 1) exactly if y is left of x. In what follows, we handle all four cases separately, but, before doing so, we introduce some further notation.

For each $i \in [t]$ and each $u \in X_i$, we define the *tail of u from X_i*, denoted $T(u, X_i)$, as the set of all points $v \in X$ with the property that every path in the cover graph G of P starting at v and ending at a point of X_1 contains u. Note that, in particular, $u \in T(u, X_i)$ and $T(u, X_i)$ is a set of consecutive elements in each linear order $L_j \in \mathcal{F}_3$.

We put $\mathrm{Cut}(x, y)$ for the set of cut vertices in the cover graph G of P which are on every path from x to y in G. Since there is no $i \in [t]$ for which $x, y \in Z_i$, $\mathrm{Cut}(x, y)$ is clearly nonempty. Let $i = i(x, y)$ denote the least $j \geq 1$ such that $\mathrm{Cut}(x, y) \cap X_j \neq \emptyset$, $u = u(x, y)$ the element of $\mathrm{Cut}(x, y) \cap Z_i$ which is closest to x in G, while $v = v(x, y)$ the element of $\mathrm{Cut}(x, y) \cap Z_i$ which is closest to y in G.

Case "x below y" Suppose we learned from the family $\mathcal{F}_5$ that x is below y and let

$$S = \{(x, y) \in \mathrm{Inc}(P) \ : \ x \text{ is below } y \text{ and } v = v(x, y) \not< y \text{ in } P\}.$$

We claim that S is reversible. Indeed, suppose to the contrary that S is not reversible, hence it must contain some alternating cycle $\{(x_\alpha, y_\alpha) \ : \ \alpha \in [k]\}$. For $\alpha \in [k]$ we have $x_\alpha \leq y_{\alpha+1}$ and $v_{\alpha+1} = v(x_{\alpha+1}, y_{\alpha+1}) \not\leq y_{\alpha+1}$, which is possible only if $x_{\alpha+1}$ is below x_α. Clearly, this statement cannot hold for all $\alpha \in [k]$ (cyclically).

Let M be the linear extension reversing S. We clearly have $x \not< y$ unless the bit for M is 1, which we assume from now on. In this case x and $v = v(x, y)$ must necessarily be different, otherwise we would get a contradiction either with Property 2 (if x and

y are comparable) or with the fact that the bit corresponding to M is 1 (if x and y are incomparable). As x is below y, we have $i_x = i(x, y)$. Then, as $y \in T(v, X_{i_x})$ and $T(v, X_{i_x})$ is a set of consecutive elements (not containing x) in each of the L_i's, we have that the relative position of x and y in L_i is always the same as that of x and v, i.e., $q(x, v, \mathcal{F}_3) = q(x, y, \mathcal{F}_3)$. On the other hand, using the family $\mathcal{F}_2$ we already learned what τ_{i_x} is, and we also know that the restriction of $\mathcal{F}_3$ to X_{i_x} is $\mathcal{B}_{i_x}$. Hence we can apply τ_{i_x} to the bit string $q(x, y, \mathcal{F}_3) = q(x, v, \mathcal{F}_3) = q(x, v, \mathcal{B}_{i_x})$ to decide whether $x < v$ or not. If not, we clearly also have $x \not< y$. On the other hand, we claim that if we arrive at $x < v$, then this already implies $x < y$. Indeed suppose to the contrary that $x \not< y$. As M is a linear extension and the corresponding bit is 1, we know that $y \not< x$, hence necessarily $x \parallel y$. However, as (x, y) was not reversed by M we must have $v < y$ and hence $x < v < y$.

Case "y below x" This case can clearly be handled in a symmetric manner involving some other linear extension M'.

To cover these two cases we add to $\mathcal{B}$ the family $\mathcal{F}_6 = \{M, M'\}$.

Case "x left of y" Suppose we learned from the family $\mathcal{F}_5$ that x is left of y. This in particular implies that $x \neq u(x, y)$ and $y \neq v(x, y)$.

Let $I(P)$ denote the set of all pairs (x, y) which satisfy Property 1 and 2, with x left of y and $x \parallel y$ in P. Using the linear order L_1 from $\mathcal{F}_2$, we define four subsets R_1, R_2, R_3, R_4 of $I(P)$ as follows:

(1) R_1 consists of all pairs $(x, y) \in I(P)$ such that $u(x, y) \leq v(x, y)$ in L_1 and $x \not< u(x, y)$.
(2) R_2 consists of all pairs $(x, y) \in I(P)$ such that $u(x, y) \leq v(x, y)$ in L_1 and $v(x, y) \not< y$.
(3) R_3 consists of all pairs $(x, y) \in I(P)$ such that $u(x, y) \geq v(x, y)$ in L_1 and $x \not< u(x, y)$.
(4) R_4 consists of all pairs $(x, y) \in I(P)$ such that $u(x, y) \geq v(x, y)$ in L_1 and $v(x, y) \not< y$.

We claim that each set in $\{R_1, R_2, R_3, R_4\}$ is reversible. We give the argument for R_1, as it is clear that the reasoning for the other three cases is symmetric. Suppose to the contrary that R_1 is not reversible and hence it contains an alternating cycle $\{(x_\alpha, y_\alpha) : \alpha \in [k]\}$. Let $\alpha \in [k]$ and $i_\alpha = i(x_\alpha, y_\alpha)$, $u_\alpha = u(x_\alpha, y_\alpha)$, $v_\alpha = v(x_\alpha, y_\alpha)$. Recall that x_α is left of y_α and $x_\alpha \not< u_\alpha$ hence we can have $x_\alpha \leq y_{\alpha+1}$ in P only if $y_{\alpha+1}$ is also left of y_α. Clearly, this statement cannot hold for all $\alpha \in [k]$ (cyclically).

For $j \in [4]$ let then N_j be a linear extensions of P that reverses all pairs in R_j and set $\mathcal{F}_7 = \{N_1, N_2, N_3, N_4\}$. Then given a pair (x, y), we conclude that $x \not< y$ in P unless the bits for the linear orders in $\mathcal{F}_7$ are $(1, 1, 1, 1)$. So for the balance of the argument, we restrict our attention to pairs (x, y) which also satisfy the following property:

Property 3 The bits for the linear orders in $\mathcal{F}_7$ are $(1, 1, 1, 1)$.

Let (x, y) be a pair satisfying Property 1 through 3, and let $i = i(x, y)$, $u = u(x, y)$ and $v = v(x, y)$. We claim that the properties enforced on (x, y) imply that $x < u$, $v < y$ in P and $u \neq v$. We give the argument for $x < u$, the reasoning for $v < y$ is clearly symmetric. Suppose to the contrary that $x \not< u$ and hence $x \not< y$. As the linear orders in $\mathcal{F}_7$ are linear extensions and the corresponding bits are $(1, 1, 1, 1)$, we know that $y \not< x$, so necessarily we have $x \parallel y$. However then (x, y) is either in R_1 or in R_3 and hence has to be reversed by either N_1 or N_3 contradicting Property 3. Finally to see that $u \neq v$ just note that otherwise we would have $x < u = v < y$ and hence $x < y$ in Q which would contradict Property 2.

As a consequence we have $x < y$ in P if and only if $u < v$ in P. As x is left of y, we clearly have $x \in T(u, X_i)$ and $y \in T(v, X_i)$. Moreover, as $u \neq v$, $T(u, X_i)$ and $T(v, X_i)$ are disjoint intervals in each of the linear orders in $\mathcal{F}_3$, therefore $q(x, y, \mathcal{F}_3) = q(u, v, \mathcal{F}_3)$. Also, by the properties of $\mathcal{F}_3$, we have that $q(u, v, \mathcal{F}_3) = q(u, v, \mathcal{B}_i)$ and hence we can determine whether $u < v$ in P by applying τ_i to these bits. The rub in these observations is that, in general, we do not have any apparent method for determining τ_i. Accordingly, our goal for the remainder of the argument is to work around this difficulty.

Given any $a \in X$, we define a uniquely determined pair $(\sigma_1(a), \sigma_2(a))$ by the following rule. For $\sigma_1(a)$, we consider sequences of the form $W = (w_0, w_1, \ldots, w_m)$ where

(1) $w_0 = a$ and
(2) if $0 \leq j < m$ and $w_j \in Z_i$, then $w_{j+1} = \rho(B_i)$ and $w_j < w_{j+1}$ in P.

Among all such sequences, it is easy to see that there is a largest non-negative integer m and a uniquely determined element $v \in X$ for which there is a sequence of this form with $w_m = v$. We then set $\sigma_1(a) = v$. When $a \neq \sigma_1(a)$, we have $a < \sigma_1(a)$ in P. The definition for $\sigma_2(a)$ is symmetric and when $a \neq \sigma_2(a)$, we have $a > \sigma_2(a)$.

Now we define two further colorings $\phi_3, \phi_4 : X \to \mathbb{T}$ as follows. For $a \in X$ we put $\phi_3(a) = \tau_i$ and $\phi_4(a) = \tau_j$ if $\sigma_1(a) \in Z_i$ and $\sigma_2(a) \in Z_j$. Let $\mathcal{F}_8$ and $\mathcal{F}_9$ be the families of $4 \cdot 2^d$ linear orders guaranteed by Lemma 2 so that given a pair (x, y), we can determine $(\phi_3(x), \phi_3(y))$ and $(\phi_4(x), \phi_4(y))$ by looking at the bits for the family $\mathcal{F}_8$ and $\mathcal{F}_9$, respectively.

For each $i \in [t]$, let us fix an arbitrary linear extension L_0^i of B_i. Then for every subset $\mathbb{S}$ of $\mathbb{T}$, we define a poset $Q(\mathbb{S})$ with ground set X by describing its cover relations. A point a is covered by a point b in $Q(\mathbb{S})$ if either of the following conditions are satisfied:

(1) There is $i \in [t]$ such that $a, b \in X_i$, $\tau_i \notin \mathbb{S}$ and the root digraph contains an edge from a to b.
(2) There is $i \in [t]$ such that $a, b \in X_i$, $\tau_i \in \mathbb{S}$ and a is covered by b in L_0^i.

Lemma 3 *For every* $\mathbb{S} \subseteq \mathbb{T}$*,* $\dim(Q(\mathbb{S})) \leq 3$.

Proof By the result of Trotter and Moore [9] mentioned earlier, to prove that $\dim(Q(\mathbb{S})) \leq 3$ it is enough to show that the cover graph of $Q(\mathbb{S})$ is a forest. This is easily seen, as for $i \in [t]$, the restriction of the cover graph of $Q(\mathbb{S})$ to X_i is either a star centered at $\rho(B_i)$ (if $\tau_i \notin \mathbb{S}$) or a path of length $|X_i|$ (if $\tau \in \mathbb{S}$). □

A classic result of Rényi [6] says that given a finite set A there is always a family $\mathcal{A}$ of $\lceil \log_2 |A| \rceil$ subsets of A that separates every pair of elements, i.e., for every distinct $a, b \in A$ there is some set in $\mathcal{A}$ which contains exactly one of them. By adding the complement of every set in $\mathcal{A}$ we arrive at a family $\mathcal{A}'$ of size $2\lceil \log_2 |A| \rceil$ with the property that for every ordered pair of distinct elements $(a, b) \in A^2$ there is some set in $\mathcal{A}'$ which contains a but does not contain b. By applying this result to $A = \mathbb{T}$, fix a family $\mathcal{S} = \{\mathbb{S}_1, \mathbb{S}_2, \ldots, \mathbb{S}_m\}$ of subsets of $\mathbb{T}$ of size $m = 2 \cdot 2^d$, so that for every ordered pair $(\tau_\alpha, \tau_\beta)$ of distinct elements of $\mathbb{T}$, there is some $j \in [m]$ such that $\tau_\alpha \in \mathbb{S}_j$ and $\tau_\beta \notin \mathbb{S}_j$. For each $j \in [m]$ also fix a realizer of size 3 for the poset $Q(\mathbb{S}_j)$, guaranteed by Lemma 3, and let $\mathcal{F}_{10}$ be the union of all these $2 \cdot 2^d$ realizers. They enable us to determine the relation of any x and y in $Q(\mathbb{S}_j)$ for each $j \in [m]$.

Now let (x, y) be again a pair satisfying Property 1 through 3 with $i = i(x, y)$, $u = u(x, y)$, $v = v(x, y)$, $\tau_\alpha = \phi_3(x)$ and $\tau_\beta = \phi_4(y)$. Further let W_x and W_y be the sequences witnessing $\sigma_1(x)$ and $\sigma_2(y)$. As $x < u$ and $v < y$, we clearly have that u and v are in the sequences W_x and W_y, respectively. Furthermore, we also have that $\rho(B_i)$ is in at most one of the sequences W_x and W_y, as otherwise we would have $x < y$ in Q, which would contradict Property 2. This clearly implies that at least one of the following statements holds:

(1) $\sigma_1(x) = u$ and so $\tau_\alpha = \tau_i$.
(2) $\sigma_2(y) = v$ and so $\tau_\beta = \tau_i$.

If $\tau_\alpha = \tau_\beta$, then necessarily $\tau_\alpha = \tau_\beta = \tau_i$, so the answer as to whether $x < y$ in P is given by applying the truth-function τ_i to the bits for the linear orders in $\mathcal{F}_3$. Therefore it remains to consider the case where $\tau_\alpha \neq \tau_\beta$.

Using the properties of $\mathcal{S}$, let j_1 and j_2 be distinct integers in $[m]$ such that τ_α belongs to $\mathbb{S}_{j_1}$ but not to $\mathbb{S}_{j_2}$, while τ_β belongs to $\mathbb{S}_{j_2}$ but not to $\mathbb{S}_{j_1}$. Then, by the definition of the posets $Q(\mathbb{S}_{j_1})$ and $Q(\mathbb{S}_{j_2})$, if $\sigma_1(x) = u$ and $\tau_\alpha = \tau_i$ then we have $x \parallel y$ in $Q(\mathbb{S}_{j_2})$, while if $\sigma_2(y) = v$ and $\tau_\beta = \tau_i$ then we have $x \parallel y$ in $Q(\mathbb{S}_{j_1})$. If from $\mathcal{F}_{10}$ we learn that $x \parallel y$ both in $Q(\mathbb{S}_{j_1})$ and in $Q(\mathbb{S}_{j_2})$ then $u \not< v$ in the linear extension L_0^i and so we conclude $x \not< y$ in P. Therefore, we may assume that $x \parallel y$ in only one of them. In this case this property also identifies in which of the previous two cases we are in, i.e., whether we have $\tau_i = \tau_\alpha$ or $\tau_i = \tau_\beta$. Then we may apply this truth function to the bits for the linear orders in $\mathcal{F}_3$ to learn whether x is less than y in P.

Case "y left of x" This case can clearly be handled in a symmetric manner. As far as the linear orders involved are considered, we may reuse the families $\mathcal{F}_8, \mathcal{F}_9, \mathcal{F}_{10}$ from the previous case, but we need to replace $\mathcal{F}_7$ with a new, but analogous family $\mathcal{F}_{11}$ of size 4.

This finishes the description of the families of linear orders and hence the proof of Theorem 2 for connected posets.

To extend the preceding proof to disconnected posets, we simply add at the beginning two linear orders, guaranteed by Lemma 1, to detect for each pair (x, y) whether x and y belong to the same component. Afterwards, we apply the construction given in the proof to each component. The manner in which the linear orders on the components are merged is arbitrary.

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