A DECOMPOSITION THEOREM FOR COLLECTIONS OF UNIVERSAL SUBCONTINUA

BY

WILLIAM T. TROTTER, JR. (CHARLESTON, S. C.)

Let X be a topological space and $A \subset X$ be a continuum. Then A is called a *universal subcontinuum* (USC) of X if $A \cap B$ is connected for every continuum $B \subset X$. In this paper, we will be concerned with the problem of decomposing a collection of USC's into subcollections which have the finite intersection property. This problem was inspired by Hadwiger and DeBrunner's extension [3] of Helly's theorem [4] for collections of compact convex subsets of a Euclidean space. Another result which is similar in statement to, but not directly related to, the main theorem of this paper is Dilworth's theorem [1]. We note that universal subcontinua were originally studied by Wallace [5] under the title "semi--chains".

Gray [2] has essentially shown that an analogous result to Helly's theorem holds for a collection of USC's in a Hausdorff space.

THEOREM 1 (Gray). *Let a be a collection of USC's in a Hausdorff space. Then a has the finite intersection property if and only if every pair of elements of a has a common point.*

We will prove that the following result is valid:

THEOREM 2. *Let a be a collection of USC's in a Hausdorff space. Suppose there exists integers p, q with $p \geqslant q \geqslant 2$ such that for every p elements of a, at least q of them have a common point. Then there exists an integer $t \leqslant p-q+1$ and a decomposition $a = a_1 \cup a_2 \cup \ldots \cup a_t$ where each a_i has the finite intersection property.*

LEMMA 1 (Wallace [5]). *The intersection of an arbitrary collection of USC's is a USC.*

LEMMA 2 (Wallace [5]). *Let A and B be USC's. If $A \cap B \neq \emptyset$, then $A \cup B$ is a USC.*

LEMMA 3. *Let $F_1, F_2, \ldots, F_n$ be non-empty pairwise disjoint USC's. Let $G_1, G_2, \ldots, G_n$ be non-empty USC's each of which intersects at least two distinct F_i's. Then $\{G_1, G_2, \ldots, G_n\}$ is not pairwise disjoint.*

Proof. The proof is by induction. Consider the case $n = 2$ and suppose $\{G_1, G_2\}$ is pairwise disjoint. Then by Lemma 2, $F_1 \cup G_1 \cup F_2$ and $F_1 \cup G_2 \cup F_2$ are USC's and, by Lemma 1, $(F_1 \cup G_1 \cup F_2) \cap (F_1 \cup G_2 \cup F_2)$ is also a USC. But $(F_1 \cup G_1 \cup F_2) \cap (F_1 \cup G_2 \cup F_2) = F_1 \cup F_2 = F_1|F_2$. The contradiction shows that $\{G_1, G_2\}$ is not pairwise disjoint.

We assume the lemma is valid for $n = 2, 3, \ldots, k-1$; $k \geqslant 3$ and consider the case $n = k$. Suppose $\{F_1, F_2, \ldots, F_k\}$ and $\{G_1, G_2, \ldots, G_k\}$ are pairwise disjoint and each G_j intersects at least two distinct F_i's. Suppose there is an F_i for which $F_i \cap G_j = \emptyset$ for every j. Then the collections $\{F_1, \ldots, \hat{F}_i, \ldots, F_k\}$ and $\{G_1, \ldots, \hat{G}_i, \ldots, G_k\}$ satisfy the hypothesis for $n = k-1$. Hence $\{G_1, G_2, \ldots, G_k\}$ is not pairwise disjoint. The contradiction shows that for every F_i, there is some G_j for which $F_i \cap G_j \neq \emptyset$.

Now suppose that for some F_i, there is only one G_j for which $F_i \cap G_j \neq \emptyset$. Then the collections $\{F_1, \ldots, \hat{F}_i, \ldots, F_k\}$ and $\{G_1, \ldots, \hat{G}_j, \ldots, G_k\}$ satisfy the hypothesis for $n = k-1$. The contradiction shows each F_i intersects at least two distinct G_j's.

We may suppose that F_1 and F_2 are intersected by G_1. Suppose we have renumbered the F_i's and G_j's such that F_{i-1} and F_i are intersected by G_{i-1}, $i = 2, 3, \ldots, p$, where $1 < p < n$. Since $F_p \cap G_{p-1} \neq \emptyset$ and F_{p-1} intersects G_{p-2} and G_{p-1}, we must have $F_p \cap G_{p-2} = \emptyset$ if $p > 2$. Then by induction, $F_p \cap G_i = \emptyset$, $i \leqslant p-2$. Since F_p meets two of the G_j, we may assume $F_p \cap G_p \neq \emptyset$. We now find that $G_p \cap F_i = \emptyset$, $i \leqslant p-1$. Since G_p meets two of the F_i, we may assume that $G_p \cap F_{p+1} \neq \emptyset$. This proves that we may assume that G_i intersects F_i and F_{i+1}, $i = 1, 2, \ldots, n-1$. Suppose G_n meets F_j, $j < n$. Since F_j and F_{j+1} are intersected by G_j, $G_n \cap F_{j+1} = \emptyset$. Then by induction, $G_n \cap F_i = \emptyset$, $i > j$. Likewise $G_n \cap F_i = \emptyset$, $i < j$. This contradiction proves the lemma.

THEOREM 3. *Let a be a collection of non-empty USC's of a Hausdorff space. Suppose there is an integer $n \geqslant 2$ such that for every n elements of a, at least two have a common point. Then there is an integer $t \leqslant n-1$ and a decomposition $a = a_1 \cup a_2 \cup \ldots \cup a_t$ where each a_i has the finite intersection property.*

Proof. The proof is by induction. Theorem 1 shows that the theorem is true for $n = 2$. Suppose that the theorem is true for $n = 2, 3, \ldots, k-1$; $k \geqslant 3$ and assume a is a collection of USC's satisfying the hypothesis for $n = k$. We assume further that a does not have the required decomposition.

Then a has a subcollection $\{F_1, F_2, \ldots, F_{k-1}\}$ which is pairwise disjoint; otherwise a satisfies the hypothesis for $n = k-1$. We make the following definitions:

(a) Let γ be a subcollection of a. We will use the notation $\bigcap \gamma$ to indicate $\bigcap_{H \in \gamma} H$.

(b) For $i \leqslant k-1$, let $F_i^0 = \{F_i\}$.
For $i \leqslant k-1$, $p \geqslant 1$ define F_i^p by

$$F_i^p = \{H \in a \mid H \cap [\bigcap F_j^{p-1}] \neq \emptyset \text{ if and only if } i = j\} \cup F_i^{p-1}.$$

(c) For $i \leqslant k-1$, define F_i^∞ by $F_i^\infty = \bigcup_{p=0}^{\infty} F_i^p$.

We now prove that the following statements are valid:

(s_1) *For every* $i \leqslant k-1$, $F_i^0 \subset F_i^1 \subset F_i^2 \subset \ldots$

(s_2) *For every* $i \leqslant k-1$, $\bigcap F_i^0 \supset \bigcap F_i^1 \supset \bigcap F_i^2 \supset \ldots$

(s_3) *For every* $i \leqslant k-1$ *and every* $p \geqslant 0$, F_i^p *has the finite intersection property.*

(s_4) *For every* $i \leqslant k-1$, F_i^∞ *has the finite intersection property.*

(s_5) *For every* $H \in a$, *there is some* $i \leqslant k-1$ *such that* $H \cap [\bigcap F_i^\infty] \neq \emptyset$.

(s_6) *Let* $H \in a$; *suppose there exists* $i \leqslant k-1$ *such that* $H \cap [\bigcap F_j^\infty] \neq \emptyset$ *if and only if* $i = j$. *Then* $H \in F_i^\infty$.

Proof of (s_1). This is an immediate consequence of the definitions.

Proof of (s_2). This is an obvious consequence of (s_1).

Proof of (s_3). The proof is by induction. For each i, F_i^0 consists of a single non-empty USC and thus has the finite intersection property. Let i be fixed and consider $H, G \in F_i^1$. Then the subcollection $\{H, G, F_1, F_2, \ldots, \hat{F}_i, \ldots, F_{k-1}\}$ is a collection of k elements and thus $H \cap G \neq \emptyset$, i.e. F_i^1 has the finite intersection property. Suppose F_i^p has the finite intersection property for $p = 0, 1, 2, \ldots, m-1$; $m \geqslant 2$. Suppose $H, G \in F_i^m$ such that $H \cap G = \emptyset$. Let

$$L_0 = \{j \leqslant k-1 \mid j \neq i \text{ and } H \cap F_j \neq \emptyset\},$$

$$K_0 = \{j \leqslant k-1 \mid j \neq i \text{ and } G \cap F_j \neq \emptyset\}.$$

If $L_0 = \emptyset$, then $H \in F_i^1$ and $G \cap H \neq \emptyset$. Similarly, if $K_0 = \emptyset$, then $G \in F_i^1$ and $G \cap H \neq \emptyset$. Now suppose $L_0 \cap K_0 \neq \emptyset$ and let $j \in L_0 \cap K_0$. Then F_i and F_j are disjoint USC's intersected by G and H and thus $G \cap H \neq \emptyset$. The contradiction shows $L_0 \cap K_0 = \emptyset$.

Since $H, G \in F_i^m$, for every $j \neq i$ we have

$$H \cap [\bigcap F_j^{m-1}] = \emptyset = G \cap [\bigcap F_j^{m-1}].$$

Then for every $j \in L_0$ there is some integer q_j with $0 \leqslant q_j \leqslant m-2$ such that

$$H \cap [\bigcap F_j^{q_j}] \neq \emptyset = H \cap [\bigcap F_j^{q_j+1}].$$

Similarly, for every $j \in K_0$ there is some integer q_j with $0 \leqslant q_j \leqslant m-2$ such that

$$G \cap [\bigcap F_j^{q_j}] \neq \emptyset = G \cap [\bigcap F_j^{q_j+1}].$$

Therefore for every $j \in L_0$, there exists $A_j^0 \in F_j^{q_j+1}$ such that $H \cap A_j^0 = \emptyset$. Similarly for every $j \in K_0$, there exists $B_j^0 \in F_j^{q_j+1}$ such that $G \cap B_j = \emptyset$.

We now show that the collection $\{H, G\} \cup \{A_j^0 | \; j \in L_0\} \cup \{B_j^0 | \; j \in K_0\}$ is pairwise disjoint. We already have:

(t_1) $H \cap G = \emptyset$.

(t_2) $H \cap A_j^0 = \emptyset$ *for every* $j \in L_0$.

(t_3) $G \cap B_j^0 = \emptyset$ *for every* $j \in K_0$.

It remains to show that:

(t_4) $G \cap A_j^0 = \emptyset$ *for every* $j \in L_0$.

(t_5) $H \cap B_j^0 = \emptyset$ *for every* $j \in K_0$.

(t_6) $A_{j_1}^0 \cap A_{j_2}^0 = \emptyset$ *for every* $j_1, j_2 \in L_0$ *with* $j_1 \neq j_2$.

(t_7) $B_{j_1}^0 \cap B_{j_2}^0 = \emptyset$ *for every* $j_1, j_2 \in K_0$ *with* $j_1 \neq j_2$.

(t_8) $A_{j_1}^0 \cap B_{j_2}^0 = \emptyset$ *for every* $j_1 \in L_0$ *and for every* $j_2 \in K_0$.

Suppose $G \cap A_j^0 \neq \emptyset$ for some $j \in L_0$. Since $A_j^0 \in F_j^{q_j+1}$,

$$A_j^0 \cap [\bigcap F_j^{q_j}] \neq \emptyset = A_j^0 \cap [\bigcap F_i^{q_j}].$$

Therefore $A_j^0 \cup [\bigcap F_j^{q_j}]$ and $\bigcap F_i^{q_j}$ are disjoint USC's intersected by H and G. The contradiction proves (t_4). The proof of (t_5) is similar.

Now suppose $j_1, j_2 \in L_0, j_1 \neq j_2$ and $A_{j_1}^0 \cap A_{j_2}^0 \neq \emptyset$. Then $A_{j_1}^0 \cup A_{j_2}^0$ is a USC and $H \cap [A_{j_1}^0 \cup A_{j_2}^0] = \emptyset$. But $\bigcap F_{j_1}^{q_{j_1}}$ and $\bigcap F_{j_2}^{q_{j_2}}$ are disjoint USC's intersected by H and $A_{j_1}^0 \cup A_{j_2}^0$. The contradiction proves (t_6). The proof of (t_7) is similar.

Suppose $j_1 \in L_0$, $j_2 \in K_0$ and $A_{j_1}^0 \cap B_{j_2}^0 \neq \emptyset$. Then $\{A_{j_1}^0 \cup [\bigcap F_{j_1}^{q_{j_1}}]\} \cup \{B_{j_2}^0 \cup [\bigcap F_{j_2}^{q_{j_2}}]\}$ and $\bigcap F_i^{m-1}$ are disjoint USC's intersected by H and G. The contradiction proves (t_8).

Let H_1 and G_1 be defined by

$$H_1 = H \cup \bigcup_{j \in L_0} \{A_j^0 \cup [\bigcap F_j^{q_j}]\},$$

$$G_1 = G \cup \bigcup_{j \in K_0} \{B_j^0 \cup [\bigcap F_j^{q_j}]\}.$$

Then it is obvious that H_1 and G_1 are USC's. We prove that they are disjoint. We need to show:

(t_9) $A_{j_1}^0 \cap [\bigcap F_{j_2}^{q_{j_2}}] = \emptyset$ *for every* $j_1 \in L_0$ *and for every* $j_2 \in K_0$.

(t_{10}) $B_{j_1}^0 \cap [\bigcap F_{j_2}^{q_{j_2}}] = \emptyset$ *for every* $j_1 \in K_0$ *and for every* $j_2 \in L_0$.

Suppose $j_1 \in L_0, j_2 \in K_0$ and $A_{j_1}^0 \cap [\bigcap F_{j_2}^{q_{j_2}}] \neq \emptyset$. Then $F_{j_1} \cup A_{j_1}^0 \cup F_{j_2}$ and F_i are disjoint USC's intersected by G and H. The contradiction proves (t_9). The proof of (t_{10}) is similar. Thus $H_1 \cap G_1 = \emptyset$.

Define L_1 and K_1 as follows:

$$L_1 = \{j \leqslant k-1 | \; j \neq i \text{ and } H_1 \cap F_j \neq \emptyset\},$$

$$K_1 = \{j \leqslant k-1 | \; j \neq i \text{ and } G_1 \cap F_j \neq \emptyset\}.$$

It is obvious that $L_0 \subset L_1$ and $K_0 \subset K_1$. Since H_1 and G_1 are disjoint, we have $L_1 \cap K_1 = \emptyset$. Suppose $L_1 - L_0 = \emptyset = K_1 - K_0$. Then

$$\{H, G\} \cup \{A_j^0 | \ j \in L_0\} \cup \{B_j^0 | \ j \in K_0\} \cup \{F_j | \ j \notin L_0 \cup K_0, j \neq i\}$$

is a collection of k pairwise disjoint elements of α.

The contradiction shows that at least one of $L_1 - L_0$ and $K_1 - K_0$ is non-empty. We now repeat the process in the following manner.

For every $j \in L_1 - L_0$, there is some integer q_j with $0 \leqslant q_j \leqslant m-3$ such that

$$H_1 \cap [\bigcap F_j^{q_j}] \neq \emptyset = H_1 \cap [\bigcap F_1^{q_j+1}].$$

Similarly for every $j \in K_1 - K_0$ there is some integer q_j with $0 \leqslant q_j \leqslant m-3$ such that

$$G_1 \cap [\bigcap F_j^{q_j}] \neq \emptyset = G_1 \cap [\bigcap F_j^{q_j+1}].$$

Therefore for every $j \in L_1 - L_0$ there exists $A_j^1 \in F_j^{q_j+1}$ such that $H_1 \cap A_j^1 = \emptyset$. Similarly, for every $j \in K_1 - K_0$ there exists $B_j^1 \in F_j^{q_j+1}$ such that $G_1 \cap B_j^1 = \emptyset$.

As before, the collection

$$\{H_1, G_1\} \cup \{A_j^1 | \ j \in L_1 - L_0\} \cup \{B_j^1 | \ j \in K_1 - K_0\}$$

is pairwise disjoint. We define

$$H_2 = H_1 \cup \Big(\bigcup_{j \in L_1 - L_0} \{A_j^1 \cup [\bigcap F_j^{q_j}]\}\Big),$$

$$G_2 = G_1 \cup \Big(\bigcup_{j \in K_1 - K_0} B_j^1 \cup [\bigcap F_j^{q_j}]\}\Big).$$

Then G_2 and H_2 are disjoint USC's. We define L_2 and K_2 by

$$L_2 = \{j \leqslant k-1 | j \neq i, H_2 \cap F_j \neq \emptyset\}$$

and

$$K_2 = \{j \leqslant k-1 | j \neq i, G_2 \cap F_j \neq \emptyset\}.$$

Then $L_0 \subset L_1 \subset L_2$; $K_0 \subset K_1 \subset K_2$; and $L_2 \cap K_2 = \emptyset$. If $L_2 - L_1 = \emptyset = K_2 - K_1$, then

$$\{H, G\} \cup \{A_j^0 | j \in L_0\} \cup \{A_j^1 | j \in L_1 - L_0\} \cup \{B_j^0 | j \in K_0\} \cup \{B_j^1 | j \in K_1 - K_0\} \cup \{F_j | j \neq i, j \notin L_1 \cup K_1\}$$

is a collection of k pairwise disjoint elements of α. The contradiction shows at least one of $L_2 - L_1$ and $K_2 - K_1$ is non-empty.

This argument may be repeated to obtain sequences L_r and K_r which satisfy the following conditions:

(a) $L_r \subset \{1, 2, \ldots, k-1\}$; $r \geqslant 0$,
(b) $K_r \subset \{1, 2, \ldots, k-1\}$; $r \geqslant 0$,
(c) $L_r \subset L_{r+1}, K_r \subset K_{r+1}$; $r \geqslant 0$,
(d) $L_r \cap K_r = \emptyset$; $r \geqslant 0$.

If $r \geqslant 0$ and $L_{r+1} - L_r = \emptyset = K_{r+1} - K_r$, then

$$\{H, G\} \cup \{A_j^0 | j \in L_0\} \cup \{A_j^t | j \in L_t - L_{t-1}; 1 \leqslant t \leqslant r\}$$
$$\cup \{B_j^0 | j \in K_0\} \cup \{B_j^t | j \in K_t - K_{t-1}; 1 \leqslant t \leqslant r\} \cup \{F_j | j \neq i, j \notin L_r \cup K_r\}$$

is a collection of k pairwise disjoint elements of a. The contradiction shows at least one of $L_{r+1} - L_r$ and $K_{r+1} - K_r$ is non-empty.

It is clear that no such sequences L_r and K_r can exist. Thus the original assumption that $H \cap G = \emptyset$ must be false, i.e. $H \cap G \neq \emptyset$ and F_i^m has the finite intersection property.

Proof of (s_4). Let $i \leqslant k-1$ and $H, G \in F_i^\infty$. Then there exist integers p, q such that $H \in F_i^p$ and $G \in F_i^q$. Then by (s_1), $H \in F_i^{p+q}$ and $G \in F_i^{p+q}$ and since F_i^{p+q} has the finite intersection property, $H \cap G \neq \emptyset$. Thus F_i^∞ has the finite intersection property.

Proof of (s_5). Let $H \in a$. Suppose $H \cap [\bigcap F_i^\infty] = \emptyset$ for every $i \leqslant k-1$. Then for each i, there exists $G_i \in F_i^\infty$ such that $H \cap G_i = \emptyset$. And for each i there exists an integer p_i such that $G_i \in F_i$. Hence $H \cap [\bigcap F_i^{p_i}] = \emptyset$ for every $i \leqslant k-1$. Let $p = \sum_{i=1}^{K-1} p_i$. Then $H \cap [\bigcap F_i^p] = \emptyset$ for every $i \leqslant k-1$. We now show that for every $H \in a$ and for every integer $p \geqslant 0$, there is some $i \leqslant k-1$ such that $H \cap [\bigcap F_i^p] \neq \emptyset$.

The remainder of the proof of (s_5) is similar to the argument used in proving (s_3) and thus will be omitted.

Proof of (s_6). For every $j \neq i$, there exists $A_j \in F_j^\infty$ such that $H \cap A_j = \emptyset$. Then for each $j \neq i$, there exists an integer $p_j \geqslant 0$ such that $A_j \in F_j^{p_j}$. Let $p = \sum_{j \neq i} p_j$. Then $H \cap [\bigcap F_j^p] = \emptyset$ for each $j \neq i$ and thus $H \in F_i^{p+1}$ and hence $H \in F_i^\infty$.

We now show that $a - F_1^\infty$ satisfies the hypothesis for $n = k-1$, i.e. for every $k-1$ elements of $a - F_1^\infty$, at least two have a common point. Let $A_1, A_2, \ldots, A_{k-1} \in a - F_1^\infty$. Suppose that $\{A_1, A_2, \ldots, A_{k-1}\}$ is pairwise disjoint.

Let $L = \{i \leqslant k-1 |$ There exists $j_i \leqslant k-1$ such that $A_i \in F_{j_i}^\infty\}$ and if $L \neq \emptyset$, let $K = \{j_i | i \in L\}$. Suppose $L = \emptyset$. Then by (s_5), for each A_i there is some j such that $A_i \cap [\bigcap F_j^\infty] \neq \emptyset$. Then by ($s_6$) each A_i intersects at least two distinct elements of the pairwise disjoint collection $\{\bigcap F_1^\infty, \bigcap F_2^\infty, \ldots, \bigcap F_{k-1}^\infty\}$. Then by Lemma 3, the collection $\{A_1, A_2, \ldots, A_{k-1}\}$ is not pairwise disjoint. The contradiction shows $L \neq \emptyset$.

Suppose there exists $i_1, i_2 \in L$ with $i_1 \neq i_2$ such that $j_{i_1} = j_{i_2}$. Then A_{i_1} and A_{i_2} are both elements of $F_{i_1}^\infty$ and by (s_4), $A_{i_1} \cap A_{i_2} \neq \emptyset$. The contradiction shows card $L = \text{card} K$. Since each $A_i \in a - F_1^\infty$, we have $1 \notin K$, i.e. card $K \leqslant k-2$.

If we define $M = \{1, 2, \ldots, k-1\}$, we have $M - L \neq \emptyset$. Let $j \in M - L$; then A_j intersects a distinct pair of $\{\bigcap F_1^\infty, \bigcap F_2^\infty, \ldots, \bigcap F_{k-1}^\infty\}$, say $\bigcap F_{j_1}^\infty$

and $\bigcap F_{j_2}^{\infty}$ with $j_1 \neq j_2$. Then j_1 and j_2 are not in K, i.e. $\operatorname{card}(M-K) \geqslant 2$. Since $\operatorname{card} L = \operatorname{card} K$, we have $\operatorname{card}(M-L) = \operatorname{card}(M-K) \geqslant 2$. Then $\{A_i \mid i \in M-L\}$ and $\{\bigcap F_i^{\infty} \mid i \in M-K\}$ satisfy the hypothesis for Lemma 3. Hence $\{A_i \mid i \in M-L\}$ is not pairwise disjoint.

Therefore by the induction hypothesis, there is an integer $t \leqslant k-2$ and a decomposition $a - F_1^{\infty} = a_1 \cup a_2 \cup \ldots \cup a_t$ where each a_i has the finite intersection property. By (s_4), F_1^{∞} has the finite intersection property and thus $a = F_1^{\infty} \cup a_1 \cup a_2 \cup \ldots \cup a_t$ is a decomposition of a into $t+1$ subcollections each having the finite intersection property and $t+1 \leqslant (k-2)+1 = k-1$. Thus the theorem is true for $n = k$. By induction, it is true for all n.

We now return to the proof of Theorem 2. Suppose a is a collection of USC's which satisfies the hypothesis of Theorem 2. Then it is easy to see that a satisfies the hypothesis of Theorem 3 with $n = p-q+2$. Therefore Theorem 2 follows as a corollary to Theorem 3.

Furthermore, it is easy to see that the decomposition provided by Theorem 2 is minimal whenever q is maximal, i.e. a satisfies the hypothesis for p, q but not for $p, q+1$.

In a later paper, it will be shown that if a is a collection of USC's in a Hausdorff space, then a can be partitioned into a finite number of subcollections, each of which have the finite intersection property, if and only if a contains no infinite subcollection whose elements are pairwise disjoint.

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