Rec'd: **8/30/2012 12:00:39 PM**

ILL- Lending
ARIEL

University of Arizona Library
Interlibrary Loan
1510 E. University Blvd
Tucson, AZ 85721
(520) 621-6438 / (520) 621-4619 (fax)
OCLC: AZU
U of A Ariel: 150.135.45.156
askill@u.library.arizona.edu

ODYSSEY ENABLED

Borrower: RAPID:GAT

TN.#: 1099595

ILL #: -5756545

Call #: QA1 .C66 v29-30 1980-81

Location: Science-Engineering Library IN LIBRARY

Reference #:

Journal Title: Congressus numerantium.

Regular

Article Author: Maurer, Rabinovitch and Trotter

Article Title: Partially ordered sets with equal rank and dimension

Volume: 29?
Issue:
Month/Year: 1980
Pages: 627 - 637 **(scan notes and title/copyright pages for chapter requests)**

Email Address:

Shipping Address:
NEW: Main Library
Interlibrary Loan
704 Cherry Street
Atlanta, GA 30332-0900

Fax:

Paged by EC **(Initials)** 08/30/12 4pm

Reason Not Filled (check one):

- ☐ **NOS** ☐ **NFAC (GIVE REASON)**
- ☐ **LACK VOLUME/ISSUE**
- ☐ **PAGES MISSING FROM VOLUME**

University of Arizona Interlibrary Loan

ARIEL INFORMATION:

Ariel Address: 130.207.50.108

Enter Ariel Address Manually if unable to scan.
If Ariel address blank, send via email.

CONGRESSUS NUMERANTIUM

VOLUME 29

DECEMBER, 1980

WINNIPEG, CANADA

Partially Ordered Sets with Equal Rank and Dimension*

Stephen B. Maurer, Department of Mathematics, Swarthmore College, Swarthmore, PA 19081

Issie Rabinovitch, Department of Mathematics, Concordia University, Montreal, Quebec, Canada H3G 1M8

William T. Trotter, Jr., Department of Mathematics, Computer Science and Statistics, University of South Carolina, Columbia, SC 29208

Abstract

A family F of linear extensions of a partial order P is called a realizer when $\cap F = P$; a realizer F is called irredundant when $\cap G \neq P$ for any proper subfamily $G \subset F$. The dimension of P is the number of linear extensions in a smallest (hence irredundant) realizer; the rank is the number in a largest irredundant realizer. Rank is clearly equal or greater than dimension, and usually much greater. Here we determine those partial orders which have dimension equal to rank. Let $\underline{2}^n$ be the partial order of subsets of an n-set. Basically, our posets are subposets of $\underline{2}^n$ which include all the 1-sets and (n-1)-sets, as well as those posets obtained from these by stretching elements into chains.

1. Introduction

A partially ordered set (poset) is a pair (X,P) where X is a set (always finite in this paper) and P is an irreflexive, transitive, binary relation on X. We write $x \geq y$ when either $(x,y) \in P$ or $x = y$. A family $F = \{L_1, L_2, \ldots, L_k\}$ of linear extensions of P (all still defined on X) is called a realizer of P when $\cap F = P$, i.e., $(x,y) \in P$ iff $(x,y) \in L_i$ for all i. A realizer is said to be irredundant when $\cap G \neq P$ for any proper nonempty subfamily G of F.

The dimension d(P) of (X,P) is defined as the smallest number of linear extensions in any realizer of P. This old definition has both pure and applied interest. For instance,

* Originally presented at the Tenth Southeastern Conference on Combinatorics, Graph Theory and Computing (1979).

d(P) is easily seen to be the lowest number n such that the elements of X can be represented by points in R^n so that P is precisely the partial order induced by the usual partial order on n-tuples. As for an application, posets arise as preference relations in measurement theory. One psychological explanation as to how people come to their preference, and in particular, why they are sometimes indifferent, is that they actually rate items on several linear scales simultaneously (perhaps unknowingly) and only sense a preference when all the scales agree as to the better choice. Then d(P) is just the smallest number of scales necessary to "explain" a person's preference order P.

Unfortunately, dimension, though extensively studied, is extremely difficult to compute. See [7] for a good bibliography. It is easy enough to construct irredundant (i.e., minim<u>al</u>) realizers for any P, and every realizer containing d(P) linear extensions (i.e., a minim<u>um</u> realizer) is irredundant. However, not every irredundant realizer is minimum. In light of the difficulty of computing d(P) exactly, one might hope to get a good approximation by constructing <u>some</u> irredundant realizer of P.

With this approach in mind, the first two authors [1] made the following definition and asked the following question. The <u>rank</u> r(P) is the largest number of linear extensions in any irredundant realizer of P. How large can r(P) be relative to d(P)? Unfortunately, the answer is that, when $|X| = n$, r(P) can be as large as $\lfloor n^2/4 \rfloor$, while simultaneously d(P) can be 2. Therefore a different question is in order: for which types of posets is rank a good estimator of dimension? In particular, characterize those P for which r(P) = d(P). Let us call a poset satisfying this equality an <u>RD poset</u>. In this paper we characterize RD posets.

We will make use of a general theory of rank which we have presented elsewhere [2]. This theory makes the computation of

rank much easier than the computation of dimension, but by no means trivial in general. Basically, the theory says that it suffices to look at all digraphs with vertex set X and which meet certain simple properties related to P; the largest number of edges in any such graph is r(P). Thus the actual construction of realizers of P is entirely bypassed!

The organization of this paper is as follows. In the next section we introduce notation and state our characterization theorem. In Section 3 we provide (and reference) those concepts and facts from the general theory of rank which we use in the proof of our theorem, which follows in Section 4.

For other results about rank, see [2,3,4].

2. <u>Statement of the Characterization Theorem</u>

Let $\underline{n}$ denote an n-element chain and $\overline{\overline{m}}$ an m-element antichain. If (X,P) and (Y,Q) are posets, define the <u>join</u> $(X,P) \oplus (Y,Q)$ as the poset $(X \dot{\cup} Y,\ P \dot{\cup} Q \dot{\cup} (X \times Y))$, where $\dot{\cup}$ is disjoint union. Note that in $(X,P) \oplus (Y,Q)$, every point of X is greater than every point of Y. Let (X,P) be a poset and let $\{(Y_x,Q_x): x \in X\}$ be a family of posets. The <u>ordinal sum</u> $\sum_{x \in X} (Y_x,Q_x)$ is the poset (Z,R) where

$Z = \{(x,y): y \in Y_x,\ x \in X\}$, and $((x,y),(x',y')) \in R$ iff either $(x,x') \in P$, or $x = x'$ and $(y,y') \in Q_x$. In this paper we are concerned only with ordinal sums where (X,P) is a collection of sets ordered by inclusion and each (Y_x,Q_x) is a chain.

Let $\underline{2}^n$ denote the poset determined by the collection of all subsets of $\{1,2,\ldots,n\}$, ordered by inclusion. Let C_n denote the collection of all subposets of $\underline{2}^n$ containing all the 1-element and (n-1)-element subsets. (We say (Y,Q) is a <u>subposet</u> of (X,P) if $Y \subset X$ and Q consists of <u>all</u> pairs from P with elements from Y.) For $n \geq 3$, $1 \neq n-1$ and the subposet on

the collection of 1-element and (n-1)-element subsets is called the crown S_n^o. (See [5] for the more general S_n^k.) Finally, let C_n^* be the collection of all posets which can be obtained by taking ordinal sums of families of chains over posets from C_n.

We can now state our Theorem. Note first that $d(P) = 1$ implies $r(P) = 1$ trivially, i.e., every linear order is an RD poset. So we may henceforth restrict ourselves to posets with $d(P) \geq 2$.

Theorem 1. Let (X,P) be a poset and let n be any integer with $n \geq 2$. Then $d(P) = r(P) = n$ if and only if $(X,P) \in C_n^*$; except if n = 2 then (X,P) can also be any poset $\underline{m} \oplus \overline{2} \oplus \underline{p} \oplus \overline{2} \oplus \underline{q}$, with $m,p,q \geq 0$.

3. Theory of Rank

For poset (X,P), we say $x \neq y$ are incomparable, i.e., $(x,y) \in I_P = I$, iff $(x,y), (y,x) \notin P$. There is a partial order called forcing induced on I by P, namely, (x,y) forces (x',y') iff the transitive closure $Tr(P \cup \{(x,y)\})$ contains (x',y'). The set of nonforcing pairs, $N_P = N$, is the set of $(x,y) \in I$ which force no other pairs. Equivalently, $(x,y) \in N$ iff for all $z \in X$, $(z,x) \in P \Rightarrow (z,y) \in P$, and $(y,z) \in P \Rightarrow (x,z) \in P$. N, like other sets of ordered pairs on X, will be thought of as a digraph on X. The size of a digraph is the number of edges. A "path" will always mean "directed path".

It is easy to show that if (x,y) is nonforcing, then (y,x) is unforced. Moreover, since X is finite, every $(x,y) \in I$ is forced by some unforced pair. It follows easily (or see the proof of the similar Lemma 4.4 in [2]) that we have

Lemma 2. A family F of linear extensions of P is a realizer of P iff $(y,x) \in \cup F$ for every $(x,y) \in N$.

Thus we say that F realizes P if it "turns over" every $(x,y) \in N$.

A subgraph $H \subset N$ is unipathic relative to P, and is henceforth called a U_P subgraph, iff whenever there are two edge-disjoint directed paths from x to y in H, then $(x,y) \in P$. (If P is an antichain, a U_P graph is unipathic as previously defined in the literature.)

Theorem 3 (Main Theorem on Rank, Form 2, from [2]). Let P be nonlinear. Let r be the size of a largest subgraph of N which is either a cycle or acyclic U_P. Then $r = r(P)$.

Note: to show $r \geq r(P)$ is relatively easy; $r \leq r(P)$ is hard.

In rank theory, cycles are a nuisance, for a) N can have lots of cycles, but b) rarely does the size of some cycle equal r(P). To elaborate on claim a), $Y \subset X$ is said to have duplicate holdings in P iff for each $z \in X$, either $(y,z) \in P$ for all $y \in Y$ or none, and $(z,y) \in P$ for all y or none. Also, a complete two-way digraph on Y has both edges (y,y') and (y',y) for all $y,y' \in Y$. Then we have

Lemma 4 (3.9 in [2]). There is a cycle in N with vertex set $Y \Longleftrightarrow$ the restriction of N to Y is a complete two-way digraph $\Longleftrightarrow$ Y has duplicate holdings in P.

As for claim b), call P rank degenerate if every largest subgraph described in Theorem 3 is a cycle. Every rank degenerate poset must have duplicate holdings, but much more is true:

Theorem 5 (see proof of Main Theorem, Form 1, in [2]). P is rank degenerate iff P is a subposet of $\underline{n} \oplus \overline{3} \oplus \underline{m}$.

Once the small class of rank degenerate posets is removed, Theorem 3 requires that we avoid cycles. Happily, it is possible to restrict attention to a subset of N which is already acyclic. Namely, for any linear order L on X (L need not even extend P) define

$$N^* = N_P^* = \{(x,y) \in N: \ (y,x) \notin N \ \text{ or } (x,y) \in L\}.$$

It might seem that N^* depends on L, but in fact, up to isomorphism, it does not. This follows from Lemma 4. Thus Theorem 3 readily implies:

Theorem 6 (**Main Theorem**, Form 3, from [2]). Suppose P is **not rank degenerate**. Then r(P) is the size of any largest U_P **graph in** N^*.

As we show in Section 4, with two exceptions, RD posets do not have duplicate holdings, so for them Theorem 6 applies and $N = N^*$.

To prove Theorem 1, we need not so much the main Theorem(s) on rank as certain more preliminary facts about N. In addition to Lemma 4 these are:

Lemma 7 (3.5 in [2]). If $x_1, x_2, \ldots, x_n$ are the vertices in order of a directed path in $P \cup N$, and at least one edge is in P, then $(x_1, x_n) \in P$.

Lemma 8 (3.8 in [2]). $Tr(P \cup N) = P \cup N \cup \{(x,x) : x$ on a cycle in $N\}$.

By a transitive tournament in N, we mean a subdigraph which is a linear order. From Lemmas 7 and 8 we immediately get:

Lemma 9. If $(x_i, x_{i+1}) \in N$, $i = 1,2,\ldots,n-1$, and $(x_1, x_n) \in N$, then for all $1 \le i < j \le n$, $(x_i, x_j) \in N$. That is, $x_1, \ldots, x_n$ is the ordered vertex set of a transitive tournament in N.

(Note: basically the same argument proves the first implication in Lemma 4.)

Finally, we need one older result often used in studying dimension. A set $\{(x_i, y_i) : 1 \le i \le m\}$ is called a TM-cycle for P when $(x_i, y_i) \in I$ for all i, and $y_i \ge x_j$ in P iff $j = i+1 \mod m$. Then it is easy to verify (see Theorem 1 of [6] and commentary afterwards):

Theorem 10. Let (X,P) be a poset and let $S \subset I$. Then $Tr(P \cup S)$ is a partial order iff S contains no TM-cycle.

We will use Theorem 10 as follows. Suppose we find $H \subset N$ such that in the reverse set $\hat{H} = \{(y,x) : (x,y) \in H\}$, every pair of edges is on a TM-cycle. Then no two edges in H can be turned over in the same linear extension of P. Thus, by the definition of dimension, $d(P) \ge |H|$.

4. Proof of Theorem 1.

The outline of our proof is:

1) All posets listed in the Theorem are RD posets.
2) With two exceptions, RD posets have no duplicate holdings. One exception is the exceptional case of the Theorem. The other is contained in C_2^*. The non-exceptional RD posets are thus not rank degenerate and each has an acyclic N.
3) For any P, let t be the number of maximal transitive tournaments in N. Then $d(P) \le t$. If for some distinct maximal transitive tournaments T_1, T_2, there is a linear extension of $P \cup \hat{T}_1 \cup \hat{T}_2$, then $d(P) < t$. Now assume P has no duplicate holdings. Then $t \le r(P)$. If in addition some maximal transitive tournament has more than one edge, then $t < r(P)$. Thus, in any RD poset not excepted in 2), each edge of N is a maximal tournament; no two edges of N can be turned over simultaneously; and $|N| = d(P)$.
4) Let P be an RD poset with $d(P) = n$ and no duplicate holdings. Label N as $\{(a_i, b_i) : 1 \le i \le n\}$ so that we may set $A = \{a_i : 1 \le i \le n\}$, $B = \{b_i : 1 \le i \le n\}$. Then with one exception, the subposet of P on $A \cup B$ is the crown S_n^o.
5) For all P described in 4), P as a whole is as described in the Theorem.

We now proceed in turn with these parts.

1). Recall that elements in posets in $\hat{C}_n$ correspond to subsets of $\bar{n} = \{1,\ldots,n\}$. Set $\hat{i} = \bar{n} - \{i\}$. For any fixed P in C_n^*, let c_i be the element at the top of the chain corresponding to $\hat{i}$. Let d_i be the element at the bottom of the chain corresponding to i. Let $C = \{c_1, \ldots, c_n\}$, $D = \{d_1, \ldots, d_n\}$. It is straightforward to show that $N = \{(c_i, d_i) : 1 \le i \le n\}$; see Theorem 6.4 in [2] for the main details. When $n \ge 3$, the edges of N are independent; for n = 2 they may be a path or a cycle. In any event, $r(P) \le n$ by Theorem 3. On the other hand, every pair of edges in N are on a TM-cycle. So $d(P) \ge n$ by Theorem 10 and the remarks after it. So P is an RD poset.

As for $P = \underline{m} \oplus \overline{2} \oplus \underline{p} \oplus \overline{2} \oplus \underline{q}$, it is easy to see that N consists of 2 disjoint 2-cycles and, by Theorem 3, $r(P) = 2$. (Any 2 edges in N form a largest cycle or acyclic U_P subgraph.) Clearly $d(P) = 2$ also.

2). Suppose (X,P) has duplicate holdings. Let $X_1,\dots,X_k$ be the disjoint maximal duplicate holdings sets. Thus each $|X_i| \ge 2$. Pick some $x_i \in X_i$ for each i, and set $Y = X - \cup(X_i - x_i)$. Let Q be the subposet of P on Y. It is easy to see that if $d(Q) > 1$, then $d(Q) = d(P)$; and if $d(Q) = 1$, then $d(P) = 2$. It is equally easy to see that in all cases, $r(P) > r(Q)$: for let H be a largest subgraph of N_Q as in Theorem 3. H may be empty (if $d(Q) = 1$), but since Q has no duplicate holdings, H is acyclic in any event (Lemma 4). Now append to H any one edge between vertices of, say, X_1. The new graph is in N_P and is acyclic U_P. Thus $r(P) > r(Q)$.

Therefore, if P is an RD poset with duplicate holdings, $d(P) = r(P) = 2$ and $P = \underline{m}_1 \oplus \overline{n}_1 \oplus \underline{m}_2 \oplus \overline{n}_2 \oplus \cdots \oplus \underline{m}_k \oplus \overline{n}_k \oplus \underline{m}_{k+1}$, with all $m_i \ge 0$ and all $n_i \ge 1$. I.e., P is a so-called weak order. However, it is easy to check (use Theorem 3 and Lemma 4) that a nonlinear weak order has $r(P) = 2$ iff either a) $k = 1$ and $n_1 = 2$, or b) $k = 2$ and $n_1 = n_2 = 2$. (For the general formula for rank of a weak order, see Cor. 6.6 in [2].) Case a) is included in C_2^*; case b) is the exceptional case of the Theorem.

3). For any transitive tournament T in N, $P \cup \hat{T}$ always has a linear extension: it is easy to see that $\hat{T}$ does not contain any TM-cycles. Since every edge in N is in some maximal T, $d(P) \le t$ by Lemma 2. Likewise, if some $P \cup \hat{T}_1 \cup \hat{T}_2$ has a linear extension, Lemma 2 gives $d(P) \le t-1$.

Now suppose P has no duplicate holdings. Let $T_1,\dots,T_t$ be the maximal transitive tournaments. For each i, let e_i be the longest edge of T_i, that is, the edge from the first element a_i to the last element b_i. We claim that the graph

$H = \{e_1,\dots,e_t\}$ is U_P, hence $r(P) \ge t$ by Theorem 6. We show H is U_P by showing something much stronger: for every path in H with 2 or more edges, if x and y are its initial and final vertices, then $(x,y) \in P$. By Lemma 7, it suffices to prove this property for paths of length exactly 2.

So suppose $(a_i,b_i)(a_j,b_j)$ is a path, i.e., $b_i = a_j$. Then $(a_i,b_j) \in P \cup N$ by Lemma 8. We show $(a_i,b_j) \notin N$. For let $x_1 = a_i$, $x_2,\dots,x_m = b_i$ be the vertices in order of T_i, and let $x_m = a_j$, $x_{m+1},\dots,x_{m+n} = b_j$ be the vertices in order of T_j. If $(a_i,b_j) \in N$, then by Lemma 9, every $(x_s,x_{s'}) \in N$ for $s < s'$. But then $x_1,\dots,x_{m+n}$ is a bigger tournament than either T_i or T_j, contradicting maximality. (There cannot be any repeats among the x_s, for then N would contain a cycle.)

Now suppose some T, say T_1, has more than one edge, i.e., at least 3 vertices. Pick any $e_1' = (a_1,x) \ne e_1$ from T_1. We claim $H \cup e_1'$ satisfies the same condition on paths of length 2 as H, hence is U_P. Thus $r(P) \ge t + 1$. To prove this claim, we need only show the condition for paths $e_i e_1'$ and $e_1' e_i$, $i \ge 2$; for e_1 and e_1' do not form a directed path. The proof is essentially the same as the proof for H: if any such path had its end pair in N, then T_i would not be maximal.

4). If $i \ne j$, then $a_i \ne a_j$ and $b_i \ne b_j$. For otherwise e_i, e_j can be turned over together, violating $d(P) = n$ by Lemma 2. So N consists of vertex disjoint paths. No such path can contain 3 edges. For if x_1,x_2,x_3,x_4 were consecutive vertices in N, then (x_1,x_2) and (x_3,x_4) could be turned over together: a TM-cycle would imply $(x_3,x_2) \in P \cap I = \emptyset$.

Moreover, if one path in N contains 2 edges, then there are no other paths! For suppose N contained (x_1,x_2), (x_2,x_3), and (x_4,x_5). Since $d(P) = n$, Lemma 2 and Theorem 10, applied to $\{(x_2,x_1),(x_5,x_4)\}$, would imply $(x_4,x_2) \in P$, and applied to $\{(x_3,x_2),(x_5,x_4)\}$ would imply $(x_2,x_5) \in P$. Thus $(x_4,x_5) \in P \cap N = \emptyset$.

Thus either $n = 2$ and N is a 2-path (the exception), or **else N consists of** n independent edges (here too $n \geq 2$, since P is **not linear)**. In the latter case, since no e_i, e_j may be turned over **together,** Theorem 10 implies (a_i,b_j), $(a_j,b_i) \in P$. Moreover $(a_i,a_j) \in I$, else either e_i or e_j is in P. Likewise, $(b_i,b_j) \in I$. Thus the subposet of P on $A \cup B$ is the crown S_n^o. (Previously we defined the crown for $n \geq 3$ only, but when $n = 2$ the poset we now get on $A \cup B$ is a natural choice to call S_2^o; only the interpretation of vertices in terms of distinct 1-sets and (n-1)-sets is missing, and as we will see, distinctness of set lables is about to be lost anyway.)

5). For all $x \in X$, define $A(x) = \{i: a_i \geq x \text{ in } P\}$, $B(x) = \{i: x \geq b_i \text{ in } P\}$. We will show:

i) For each x, A(x), B(x) partition $\bar{n} = \{1,\ldots,n\}$.

ii) $(x,y) \in I$ iff neither B(x) nor B(y) is contained in the other.

iii) If $B(x) \subsetneq B(y)$, then $(y,x) \in P$.

It follows from ii) that for each $S \subset \bar{n}$, the restriction of P to $\{x: B(x) = S\}$ is a linear order. It then follows from iii) that P is an ordinal sum of chains, where the sum is over a subposet (Z,Q) of $\underline{2}^n$. It follows from the analysis in 4) of the subposet on $A \cup B$ that Z contains all 1-element and (n-1)-element subsets. (When $n = 2$, then $n-1 = 1$ and $A \cup B$ gives us each 1-set at least once, whether $A \cup B$ is a crown or the exception.) Thus, when we prove i) - iii), we have finished the proof of our Theorem.

We need two observations. First, for any P whatsoever, if $(x,y) \in I$, then (x,y) forces some $(z,w) \in N$, where $z \geq x$ and $y \geq w$ in P. Second, for all P as in 4), including the exception, $(a_i,a_j),(b_i,b_j) \in I$. Now to prove i), clearly no $i \in A(x) \cap B(x)$, for then $(a_i,b_i) \in P$. Suppose some $i \notin A(x) \cup B(x)$. It cannot be that both (x,a_i), $(b_i,x) \in P$, for then $(b_i,a_i) \in P$. So either $(a_i,x) \in I$ or $(x,b_i) \in I$. (a_i,x) would force some (a_j,b_j), and since then $a_j \geq a_i$ and $x \geq b_j$ in P, we would have $j = i$ and

$i \in B(x)$. Dually, (x,b_i) would force the contradiction $i \in A(x)$.

To prove ii), if $(x,y) \in I$, then (x,y) forces some (a_i,b_i). By the definition of forcing, $i \in A(x) \cap B(y) = (\bar{n} - B(x)) \cap B(y)$, so $i \in B(y) - (B(x)$. Dually, $(y,x) \in I$ too, and this forces some (a_j,b_j), so $j \in B(x) - B(y)$. To prove the reverse implication, if $(x,y) \in P$, then clearly $B(y) \subset B(x)$, and if $(y,x) \in P$, then $B(x) \subset B(y)$. These facts also prove iii), for of the exhaustive possibilities $(x,y) \in I$, $(x,y) \in P$, $(y,x) \in P$, the only one compatible with $B(x) \subsetneq B(y)$ is $(y,x) \in P$.

References

1. S. Maurer and I. Rabinovitch, "Large Minimal Realizers of a Partial Order," Proceedings AMS 66 (1978) 211-216.
2. S. Maurer, I. Rabinovitch and W. T. Trotter, "Large Minimal Realizers of a Partial Order II," Discrete Math., to appear.
3. S. Maurer, I. Rabinovitch, and W. T. Trotter, "A Generalization of Turán's Theorem," Discrete Math., to appear.
4. I. Rabinovitch and I. Rival, "The Rank of a Distributive Lattice," Discrete Mathematics 25 (1979) 275-279.
5. W. T. Trotter, "Dimension of the Crown S_n^k," Discrete Mathematics 8 (1974) 85-103.
6. W. T. Trotter and J. I. Moore, "The Dimension of Planar Posets," J. Combinatorial Theory (B) 22 (1977) 54-67.
7. W. T. Trotter and J. I. Moore, "Characterization Problems for Graphs, Partially Ordered Sets, Lattices, and Families of Sets," Discrete Mathematics 16 (1976) 361-381.