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ILLiad TN: 256280

Journal Title: The Theory and Applications of Graphs

Call #: QA166 .I55 1980

Volume:

Location: 3E - 8/30

Issue:

Month/Year: 1981

Pages: 397 - 407

Article Author: Hopkins and Trotter

Article Title: A bound on the interval number of a complete multipartite graph

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A Bound on the Interval Number of a Complete Multipartite Graph

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ABSTRACT

The *interval number* of a graph G denoted $i(G)$, is the least positive integer t for which G is the intersection graph of a family of sets each of which is the union of t pairwise disjoint intervals of the real line. For example, a graph G is an interval graph if and only if $i(G) = 1$, while $i(C_n) = 2$ for all $n \geq 4$. Griggs showed that the maximum value of the interval number of a graph on n vertices is $\lceil (n+1)/4 \rceil$ and Trotter and Harary showed that the interval number of the complete bipartite graph $K(n_1, n_2)$ is given by the formula $i(K(n_1, n_2)) = \lceil (n_1 n_2 + 1) / (n_1 + n_2) \rceil$. Several researchers have been investigating the problem of determining the interval number of complete multipartite graphs, and it was conjectured that the interval number of the complete multipartite graph $K(n_1, n_2, n_3, \dots, n_p)$, where $n_1 \geq n_2 \geq n_3 \geq \dots \geq n_p$ and $p \geq 3$, equals the interval number of the complete bipartite graph $K(n_1, n_2)$. In support of this conjecture, Matthews proved that for every $p \geq 3$, if $n_1 = n_2 = n_3 = \dots = n_p$, then $i(K(n_1, n_2, \dots, n_p)) = i(K(n_1, n_2))$. However, D. West disproved the conjecture by showing that for each $n \geq 3$, there exists a constant c_n so that if $n_1 = n^2 - n - 1$, $n_2 = n_3 = \dots = n_p = n$, and $p \geq c_n$, then $i(K(n_1, n_2, n_3, \dots, n_p)) = 1 +$

$i(K(n_1, n_2))$. In view of West's counterexample, it was suggested

that the interval number of a complete multipartite graph might exceed the interval number of the bipartite graph, formed by the largest two parts, by an arbitrarily large amount. In this

paper, we prove to the contrary that $i(K(n_1, n_2, n_3, \dots, n_p)) \leq i + i(K(n_1, n_2))$ for all p, n_1, n_2, \dots, n_p with $p \geq 3$ and $n_1 \geq n_2 \geq n_3 \geq \dots \geq n_p$.

1. Introduction.

In recent years, there has been considerable interest in

generalizations of interval graphs. Much of the research is motivated by the wide range of interpretations which may be

given to optimization and extremal problems involving interval graphs. In this paper, we consider the subject of t -interval

graphs. For a positive integer t , we represent a graph as the intersection graph of a family of sets each of which is the

union of t pairwise disjoint intervals of the real line.

Among the several extremal problems involving t -interval graphs, we will be concerned with minimizing t for a given graph or

class of graphs. If we view a t -interval graph as a work schedule permitting cooperation between certain specified components

of the work force while safeguarding against interference

between other components, then the minimization of t yields a schedule in which each component has relatively few work periods.

Consequently, the inherent inefficiency of starting up and closing down unnecessary work periods of short duration and

limited productivity is avoided.

Among the classes of graphs for which this extremal problem is quite natural is the class of complete bipartite graphs where

the work force is subdivided into two units with no interference permitted between any two components in the same unit, but cooperation required between any two components from different

units. In this paper, we will to complete multipartite graph problem in the multipartite case from the bipartite graph.

2. Notation and Terminology. Trotter and Harary [4] de

of a graph G as a function $x \in G$ a sequence $F(x)(1), F(x)(2), \dots, F(x)(p)$ intervals of the real line \mathbb{R} distinct vertices, we have x if there exists a pair of inte

that $F(x)(i) \cap F(y)(j) \neq \emptyset$. $i(G)$, is then defin

for which G has a t -interval $i(G)$ is the least integer t

graph of a family of sets each t closed intervals of \mathbb{R} . I

interval graph if and only if $i(G) = t$ We will find it convenient

(degenerate) closed interval a t -interval graph if and only if $i(G) = t$.

For example, Figure 1 provides a graph G . Note that $i(G) = 2$.

Furthermore, we will delete representation, all isolated

may be simplified as in Figure 1 and 2 to intervals will be spread out

Throughout this paper, we will bear in mind that

For example, it was suggested that the multipartite graph might be a complete multipartite graph, formed by the union of a large amount. In this case, $i(K(n_1, n_2, n_3, \dots, n_p)) \leq n_1 + n_2 + \dots + n_p$ with $p \geq 3$ and

considerable interest in such of the research is representations which may be problems involving interval the subject of t -interval we represent a graph as sets each of which is the intervals of the real line. involving t -interval graphs, t for a given graph or interval graph as a work schedule to obtain specified components against interference minimization of t yields a relatively few work periods. cost of starting up and cost of short duration and

which this extremal problem is to complete bipartite graphs where units with no interference in the same unit, but components from different

units. In this paper, we will discuss the natural generalization to complete multipartite graphs and will show that the extremal problem in the multipartite case does not differ substantially from the bipartite graph.

2. Notation and Terminology.

Trotter and Harary [4] defined a t -interval representation of a graph G as a function F which assigns to each vertex $x \in G$ a sequence $F(x)(1), F(x)(2), \dots, F(x)(t)$ of closed intervals of the real line \mathbb{R} so that for every pair x, y of distinct vertices, we have x adjacent to y in G if and only if there exists a pair of integers i, j with $1 \leq i, j \leq t$ so that $F(x)(i) \cap F(y)(j) \neq \emptyset$. The *interval number* of a graph G , denoted $i(G)$, is then defined as the least positive integer t for which G has a t -interval representation. Alternately, $i(G)$ is the least integer t for which G is the intersection graph of a family of sets each of which is the union of at most t closed intervals of \mathbb{R} . In particular, a graph G is an interval graph if and only if $i(G) = 1$.

We will find it convenient to consider a point as a (degenerate) closed interval and will frequently use this convention in specifying an interval representation of a graph. For example, Figure 1 provides a 2-interval representation of the graph G . Note that G is not an interval graph so $i(G) = 2$.

Furthermore, we will delete from Figure 1 for an interval representation, all isolated intervals and points. So Figure 1 may be simplified as in Figure 2.

Throughout this paper, we will use diagrams similar to those shown in Figures 1 and 2 to illustrate interval representations. Intervals will be spread out vertically for clarity but the reader should bear in mind that all intervals are to be projected

J. Griggs and D. West [1] bound on $i(G)$ as a function of

Theorem 3 [1]. If Δ is the maximum degree in G , then $i(G) \leq \lceil (\Delta+1)/2 \rceil$.

Griggs and West [1] show that

triangle-free, then equality holds. The result follows as an immediate

Corollary 4 [1]. For each n , there exists a complete

n -cube Q_n is given by $i(Q_n) = \lceil n/2 \rceil$.

Trotter and Harary [4] determine

number of a complete bipartite graph $K(m, n)$ is given by:

Theorem 4 [4]. The interval number $i(K(m, n))$ is given by:

$$i(K(m, n)) = \lceil \min\{m, n\} \rceil$$

3. Interval Numbers of Complete Multipartite Graphs

In the remainder of this section we

the computation of the interval number of a complete multipartite graph

$K(n_1, n_2, \dots, n_p)$ will require $n_1 \geq n_2 \geq \dots \geq n_p$.

denote the interval number of $K(n_1, n_2, \dots, n_p)$ by $i(n_1, n_2, \dots, n_p)$.

note that $i(n_1, n_2, \dots, n_p) = i(n_1, n_2, \dots, n_p, 1)$.

have $i(n_1, n_2, \dots, n_p) \leq i(n_1, n_2, \dots, n_p, 1)$.

$i(n_1, n_2, \dots, n_p)$ never exceeds n_1 .

construction due to M. Matthews [5] shows that $K(n, n)$ can be

with each a_i adjacent to every other a_j for $i \neq j$. First let n be even, say $n = 2k$.

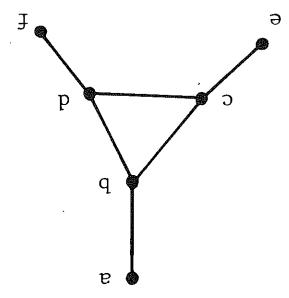


Figure 1.

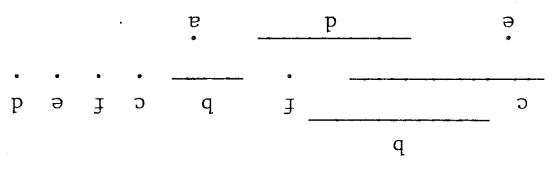


Figure 2.

onto a single horizontal line.

We now present a brief summary of recent research involving interval numbers. We begin with the following elementary result due to Trotter and Harary [4].

Theorem 1 [4]. If T is a tree, then $i(T) \leq 2$.

J. Griggs [2] has established the following upper bound on the interval number as a function of the order of the graph.

Theorem 2 [2]. If G is a graph on n vertices, then $i(G) \leq \lceil (n+1)/4 \rceil$.

here is a 3-representation (with

diagram provides an $r + 1$ - interval representation of $K(n, n)$.

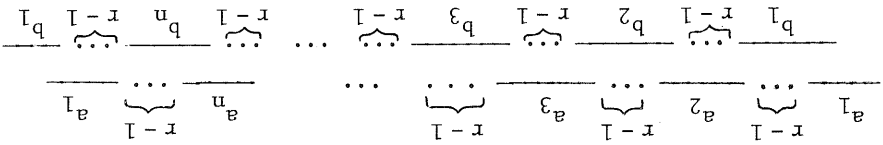


Figure 3.

In the gap between a_i and a_{i+1} (cyclically), occur the $r - 1$ points corresponding to $a_{i+2}, a_{i+3}, \dots, a_{i+r}$ (cyclically). Similarly, the gap between b_i and b_{i+1} contains points corresponding to $b_{i+2}, b_{i+3}, \dots, b_{i+r}$. Here is a diagram when $r = 3$. For simplicity, only the subscripts are given.

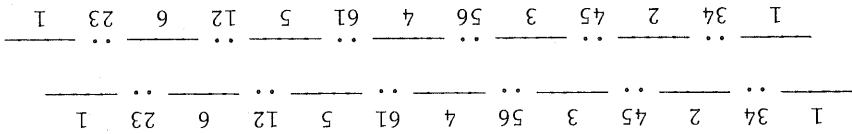


Figure 4 .

We shall continue to use the convention followed in Figures 3 and 4 for bipartite and multipartite graphs i.e., the diagram will be presented in "levels" with all intervals occurring in the same level corresponding to vertices in the same part. The reader is encouraged to compare this example with the construction given by Trotter and Harary [4] for a 4-representation of $K_{6,6}$. The advantage of Matthews' construction is that it can easily be extended to multipartite graphs. It suffices to add additional "levels" to the diagram following the same interval-section pattern as determined by the first two. For example,

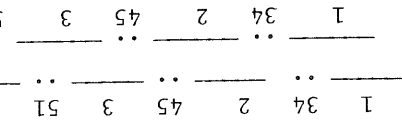


Fig 5

Reading the diagram from

each level the first occurrence resulting diagram is an $r + 1$

interval representation of K above. For example, here is the

When n is odd, say $n = r + 1$.

$i(n, n) \leq i(n_1, n_2, \dots, n_p) \leq r + 1$. It follows that when $n = 2r$,

representation of $K(n_1, n_2, \dots)$ construction produces for each

More generally, it is easy

Fig 5

