AN EXTREMAL PROBLEM IN RECURSIVE COMBINATORICS

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§0. Introduction.

In this paper, we shall prove that every recursive interval order of width w can be covered by $3w-2$ recursive chains. To establish thatthis result is best possible, we show that there exists a recursive interval order of width w that cannot be covered by $3(w-1)$ recursive chains. Before discussing the significance of these theorems, we pause to remind the reader of some of the fundamental definitions in recursive conbinatorics. A function $f:N^k \to N$, where N is the set of natural numbers, is <u>recursive</u> iff there exists an algorithm (i.e., a finite computer program) which upon input of a sequence of length k of natural numbers $\bar{n}$, outputs $f(\bar{n})$ after a finite number of steps. A subset of N^k, (i.e., a k-ary relation) is recursive provided that its characteristic function is recursive. For a more formal treatment of recursive functions and recursive relations see [R]. A partial order $\mathcal{P} = (P,\leq)$ is <u>recursive</u> provided that P is a recursive set and $\leq$ is a recursive relation. Similarly a graph $G = (V,E)$ is <u>recursive</u> provided that V is a recursive set and E is a recursive relation. G is <u>recursively</u> k-colorable iff there is a k-coloring of G that is a recursive function. A partial order $\mathcal{P} = (P,\leq)$ is an <u>interval order</u> iff $\mathcal{P}$ is isomorphic to a partial order $\mathcal{I} = (I,\leq^*)$ such that I is a set of intervals of the real line and for all $i, j \in I$, $i <^* j$ iff the right end point of i is less than the left end point of j. The incomparability graph of a interval order is an _interval graph_. P is a _recursive interval order_ iff P is a recursive partial order which is also an interval order. Recursive interval graphs are defined analogously.

One of the attractions of finite combinatorics is the explicit descriptions of the objects under consideration. These explicit descriptions are usually lost when one passes to infinite combinatorics. For example consider the case of the infinite version of Dilworth's Theorem [D]. When one proves that every infinite partial order of finite width w can be covered by w chains the proof does not provide a description of those chains, it merely demonstrates their existence. The main results of this paper are part of a program to extend the domain of finite combinatorics to recursive objects. While these objects are generally infinite, they are explicitly described by the algorithms associated with them. While we can not hope to assimilate the total amount of information about an infinite structure that is in some sense stored in the finite algorithm that describes it, we certainly have complete information about any finite part of the structure. A similar situation exists even in finite combinatorics when one tries to comprehend a very large and complicated graph, but can really only understand small subgraphs of it.

Kierstead [K] proved that every recursive partial order of width w can be covered by $\frac{5^w-1}{4}$ recursive chains and there is a recursive partial order of width w that cannot be covered by $4(w-1)$ recursive chains. The latter part of this result explains why proofs of the infinite version of Dilworth's Theorem do not give explicit descriptions of the coverings. The first part shows that not all is lost. We can still explicitly describe coverings if we are allowed to use additional chains. There is a disturbing gap between $4(w-1)$ and $\frac{5^w-1}{4}$. The main result of this paper closes that gap for the class of interval orders. The proof actually shows the stronger result that every recursive interval graph G whose clique number is $\omega(G)$ can be recursively $(3\omega(G)-2)$-colored. The recursive chromatic number of graphs is discussed in papers by Bean [B], Schmerl [S], [S1], and Kierstead [K].

Support for the thesis that recursive combinatorics is a natural extension of finite combinatorics comes from the experience that the arguments used to prove theorems have the flavor and style of arguments used in finite combinatorics. To prove that a recursive partial order can be covered by m recursive chains, one must produce an algorithm that provides these chains. Each time the algorithm makes a (irrevocable) decision about the chain into which an element of the partial order is to be inserted, it must base this decision on a finite amount of information about the partial order. The problem of constructing a recursive partial order that cannot be covered by m recursive chains requires somewhat more sophistication. One must be sure that no algorithm produces a set of m chains that covers the partial order. Lemma 1 reduces this recursion theoretical problem to a strictly combinatorial problem. The reduction requires the following definition.

Definition.0 Let $G(m,w)$ be the following infinite game for two players X and Y. At his k+1st turn, X will have constructed a finite interval order $P = (P,\le)$ of width at most w. Y will have covered this interval order by m sets. X makes his k+1st play by extending his interval order to one new element so that the width remains at most w. Y then makes his k+1st play by adding this new element to exactly one of the sets in his covering. If at the end of ω (the first infinite ordinal) plays, each of the m sets in Y's covering is a chain, then Y wins; otherwise X wins. □

Lemma 1. If X has a winning strategy for the game $G(m,w)$, then there exists a recursive interval order of width w that cannot be covered by m recursive chains.

Sketch of Proof. (See [K] for additional details.) If X has a winning strategy S for the game $G(m,w)$, then there exists a finite s such that X can always be assured of winning after s plays. Thus S is essentially finite and thus recursive. The set of algorithms is countable; moreover it can be effectively listed. Our idea is to construct an interval order with infinitely many distinct parts such that the ith algorithm does not even provide a cover of the ith part of the interval order by m chains. Roughly speaking, we do this by using the recursive strategy S to play the game $G(m,w)$ against the ith algorithm. □

The reader may notice that our proof of Theorem 5 provides a winning strategy for Y in the game $G(3w-2, w)$. The problem of which player has a winning strategy for the game $G(m,w)$ seems to have interest of its own.

Notation. Let $P = (P,\leq)$ be a partial order. The width of P is denoted by $w(P)$. If $x,y \in P$ and x is incomparable to y, we write $x||y$. If x is comparable to y we write $x <> y$. If $x \in P$, $A \subset P$, and for all $a \in A$, $a||x$ we write $x||A$. If $p \in P$ then

$$\{D(p) = \{q \in P: q < p\} \quad \text{and} \quad U(p) = \{q \in P: p < q\} .$$

Let $B \subset N$. The set $\{b \in B: b < p \text{ in } N\}$ is denoted by B^P. If $G = (V,E)$ is a graph, then the maximum size of a complete subgraph of G is denoted $\omega(G)$, and is called the clique number of G.

§1. The Main Results

We begin with some basic properties of interval orders.

Lemma 2. (Fishburn [F]): P is an interval order iff P is a partial order which does not have a suborder isomorphic to $\underline{2} + \underline{2}$. □

Definition 3. Let $\mathcal{P} = (P,\le)$ be a partial order and let $S,T \subset P$ be antichains. $S \le T$ iff for every $s \in S$ there exists $t \in T$ such that $s \le t$. □

Lemma 4. If $\mathcal{P} = (P,\le)$ is an interval order and $S,T \subset P$ are maximal antichains, then $S \le T$ or $T \le S$.

Proof. Suppose that not $S \le T$. Then for some $s \in S$ and every $t \in T$ not $s \le t$. Since T is maximal there exists $t \in T$ such that $s <> t$. Thus $t < s$. Now suppose that not $T \le S$. Then again there exist $s' \in S$ and $t' \in T$ such that $s' < t'$. Since S and T are antichains, $s' \ne s$ and $t' \ne t$. Thus $s || s'$ and $t || t'$. But then the suborder $\{s,t,t',s'\}$ is isomorphic to $\underline{2} + \underline{2}$, which contradicts the hypothesis that $\mathcal{P}$ is an interval order. □

Theorem 5. If $\mathcal{P} = (P,\le)$ is a recursive interval order of width w then P can be covered by $3w-2$ recursive chains.

Proof. We argue by induction on w. If $w = 1$ then P itself is a chain. So suppose $w = k+1$. Define B inductively by $B = \{p \in P: w(B^p \cup \{p\}) \le k\}$. B is a maximum suborder of $\mathcal{P}$ of width k and B is recursive since $\mathcal{P}$ is recursive. Let $A = P-B$. A also is recursive. By the inductive hypothesis, B can be covered by $3k-2$ recursive chains. Thus it suffices to show that A can be covered by three recursive chains. This will be accomplished by proving that each element of A is incomparable to at most two other elements of A. Then surely the greedy algorithm provides a covering of A by three recursive chains.

We begin by showing that the width of A is at most two. Consider three distinct elements $q,r,s \in A$. Then there exist antichains $Q,R,S \subset B$ of width k such that $q||Q$, $r||R$, and $s||S$. Without loss of generality $Q \le R \le S$. Suppose $r||q$ and $r||s$. We shall show that $q < s$. Since $q||r$ and $w(P) \le k+1$, there exists $r' \in R$ such that $q <> r'$. Since $q||Q$, $r' \notin Q$. Since $w(B) \le k$, there exists $q' \in Q$ such that $q' <> r'$. Since $Q \le R$, there exists $r_0 \in R$ such that $q' \le r_0$. Thus since R is an antichain $q' \le r'$. Since $q||q'$, $q \le r'$. Similarly, there exists $r'' \in R$ such that $r'' \le s$. Since P does not have any suborder isomorphic to $\underline{2} + \underline{2}$, we may choose $r' = r''$. Thus $q < s$.

Now suppose q,r,s, and t are distinct elements of A such that $q||\{r,s,t\}$. Then without loss of generality $r < s < t$. Since $s \in A$ there exists an antichain $S \subset B$ of length k such that $s||S$. Since $s||q$ and $w(P) \le k+1$, q is comparable to some element $s' \in S$. If $s' < q$, then $s'||r$ and the suborder $\{s',q,r,s\}$ is isomorphic to $\underline{2} + \underline{2}$, which contradicts the hypothesis that P is an interval order. Similarly if $q < s'$, then $s'||t$ and the suborder $\{q,s',s,t\}$ is isomorphic to $\underline{2} + \underline{2}$. □

Notice that the algorithm, implicit in the above proof, for covering P by recursive chains uses only information about the comparability of various elements of P, not their order. Thus if the incomparability graph of an interval order P is recursive, then P can be covered by $3w(P)-2$ recursive chains. Thus we have actually proved the following stronger theorem.

Theorem 6. If G is a recursive interval graph then G can be recursively $3\omega(G)-2$ colored.

Theorem 7. For every positive integer w there exists a recursive interval order of width w that cannot be covered by $3(w-1)$ recursive chains.

Proof. By Lemma 1 it suffices to show that X has a winning strategy for the game $G(3(w-1),w)$. We argue by induction on w. Clearly Y cannot win $G(0,1)$. We illustrate the inductive step of the proof by first considering the case $w = 2$. We must provide a winning strategy for X in the game $G(3,2)$.

X starts the game by creating a linear order of length $3 \cdot 3 + 1 = 10$. Y is forced to put four of these points into the same chain, say $b_1, b_2, b_3, b_4 \in C_1$ and $b_1 < b_2 < b_3 < b_4$. Now X plays two points d_1 and d_4 such that $d_1 < d_4$, $d_i \| b_i$, $U(d_i) = U(b_i)$ and $D(d_i) = D(b_i)$ for $i = 1,4$. This is illustrated in Figure A.

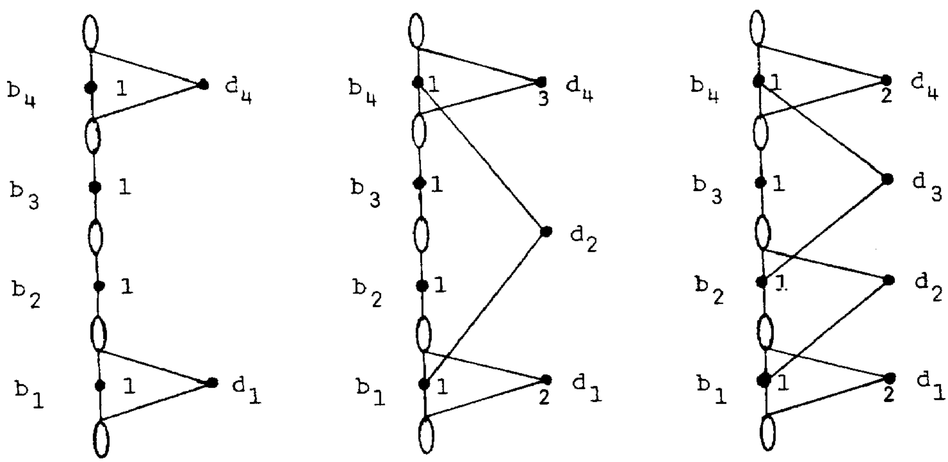

Fig. A Fig. B Fig. C

The loops represent possibly empty chains. The number next to a point is the chain into which Y has put the point. Clearly P is still an interval order of width 2. Y cannot put d_1 or d_4 into C_1. First suppose that Y puts d_1 into C_2 and d_4 into C_3 where C_2 and C_3 are distinct new chains. Then X plays d_2 such that $s < d_2$ iff $s \le b_1$ and $d_2 < t$ iff $b_4 \le t$. This is illustrated in Figure B. P still is an interval order of width 2, but Y cannot put d_2 into C_1, C_2, or C_3 since $d_2 || b_2$, $d_2 || d_1$, and $d_2 || d_4$. If instead Y decides to put d_1 and d_2 into the same new chain C_2, then X plays two more points d_2 and d_3 such that $s < d_i$ iff $s \le b_{i-1}$ and $d_i < t$ iff $b_i < t$ and not $t \le d_{i+1}$, for $i = 2,3$. This is illustrated in Figure C. Again P is an interval order of width two. Y cannot put d_2 and d_3 into the same chain since $d_2 || d_3$. Also Y cannot put either d_2 or d_3 into C_1 or C_2 since $d_i || b_i$ for $i = 2,3$, $d_2 || d_1$, and $d_3 || d_4$. Thus X wins.

Now suppose $w = k+1$. We must exhibit a winning strategy for X in the game $G(3w, w+1)$. By the inductive hypothesis X has a winning strategy S for the game $G(3(w-1),w)$. X should start the game $G(3w, w+1)$ by using S to build a chain of suborders $B_i, 0 \le i < 3\binom{3w}{3(w-1)} + 1$, such that

i) each B_i is an interval order of width w;

ii) if $x \in B_i$, $y \in B_j$, and $i < j$ then $x < y$;

iii) Y is forced to use $3w-2$ chains to cover each B_i.

There must be $3w-2$ chains $C_1, \ldots, C_{3w-2}$ and four suborders $B_{i_1}, \ldots, B_{i_4}$ such that $C_j \cap B_{i_k} \neq \emptyset$ for $1 \le j \le 3w-2$ and $1 \le k \le 4$. X plays the rest of the game as before but with the suborders B_{i_k} taking the part of the points b_k. Y will be forced to put the elements d_k into new chains and will require three new chains. However only two new chains are available. □

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References

[B] Bean, D., Effective coloration, J. Symbolic Logic 41 (1976), 469-480

[D] Dilworth, R. P., A decomposition theorem for partially ordered sets, Ann. of Math. 51 (1950) 161-166.

[F] Fishburn, P. C., Intransitive indifference with unequal indifference intervals, J. Math. Psychol. 7 (1970) 144-149.

[K] Kierstead, H. A., An effective version of Dilworth's Theorem, to appear, Trans. Amer. Math. Soc.

[K1] Kierstead, H. A., Recursive colorings of highly recursive graphs, preprint.

[R] Rodgers, H., Theory of Recursive Functions and Effective Computability, McGraw Hill (1967)

[S] Schmerl, J. H. Recursive colorings of graphs, Can J. Math., Vol. XXXII No.4 (1980) 821-830.

[S1] Schmerl, J. H. The effective version of Brooks' Theorem, preprint.