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DIMENSION THEORY FOR ORDERED SETS

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ABSTRACT

In 1930, E. Szpilrajn proved that any order relation on a set X can be extended to a linear order on X. It also follows that any order relation is the intersection of its linear extensions. B. Dushnik and E.W. Miller later defined the *dimension* of an ordered set $P = \langle X;\leq\rangle$ to be the minimum number of linear extensions whose intersection is the ordering $\leq$.

For a cardinal m, $\underset{\sim}{2}^{m}$ denotes the subsets of m, ordered by inclusion. As the notation indicates, $\underset{\sim}{2}^{m}$ is a product of 2-element chains (linearly ordered sets). Any poset $\langle X;\leq\rangle$ with $|X| \leq m$ can be embedded in $\underset{\sim}{2}^{m}$. O. Ore proved that the dimension of a poset P is the least number of chains whose product contains P as a subposet. He also showed that the product of m nontrivial chains has dimension m. In particular, $\underset{\sim}{2}^{m}$ has dimension m, a result of H. Komm. Thus, every cardinal is the dimension of some poset.

It is usually very difficult to calculate the dimension of any "standard" poset. However, dimension can be related to other parameters of a poset. For example, the dimension of a finite poset does not exceed the size of any maximal antichain. Also, T. Hiraguchi showed that any poset of dimension $d \geq 3$ has at least $2d$ elements. Moreover, any integer $\geq 2d$ is the size of some poset of dimension d.

I. Rival (ed.), Ordered Sets, 171–211.

Let d be a positive integer. A poset is *d-irreducible* if it has dimension d and removal of any element lowers its dimension. Any poset whose dimension is at least d contains a d-irreducible subposet. Although there is only one 2-irreducible poset, there are infinitely many d-irreducible posets whenever $d \geq 3$. The set of all 3-irreducible posets was independently determined by D. Kelly and W.T. Trotter, Jr. and J.I. Moore, Jr. There is a 3-irreducible poset of any size n not excluded by Hiraguchi; i.e., for any $n \geq 6$. However, R.J. Kimble, Jr. has shown that a d-irreducible poset cannot have size $2d + 1$ when $d \geq 4$. If $d \geq 4$ and $n \geq 2d$ but $n \neq 2d + 1$, then there is a d-irreducible poset of size n.

A finite poset is *planar* if its diagram can be drawn in the plane without any crossing of lines. Planar posets have arbitrary finite dimension. However, K.A. Baker showed that a finite *lattice* is planar exactly when its dimension does not exceed 2. He also showed that the completion of a poset is a lattice that has the same dimension as the poset. Baker's results and three papers of D. Kelly and I. Rival were used to obtain the list of 3-irreducible posets.

The approach of W.T. Trotter and J.I. Moore, Jr. rested on the observation of Dushnik and Miller that a poset has dimension at most 2 if and only if its incomparability graph is a comparability graph. T. Gallai's characterization of comparability graphs in terms of excluded subgraphs was then applied.

Several other connections between dimension theory for posets and graph theory have been established. For example, posets with the same comparability graph have the same dimension. C.R. Platt reduced the planarity of a finite lattice to the planarity of an *undirected* graph obtained by adding an edge to its diagram.

E. Szpilrajn [1930] showed that any order relation on a set X can be extended to a linear order on X. He also proved that any order relation is the intersection of its linear extensions. If $\mathbb{C}$ is a family of linear orders (chains) whose intersection is the order relation $\leq$, then $\mathbb{C}$ is a *realizer* of $\leq$; we also say that $\mathbb{C}$ *realizes* $\leq$. B. Dushnik and E.W. Miller [1941] defined the *dimension* of an ordered set (poset) $\langle X;\leq\rangle$ to be the minimum cardinality of a realizer of $\leq$.

In this survey, we usually deal with the case of finite dimension. For a positive integer k, any poset of dimension at least k contains a finite subposet of dimension k. (This compactness property follows from the compactness theorem of first-order logic; see Harzheim [1970] and the review by K.A. Baker.) Moreover, any finite poset has finite dimension. Consequently, most of the posets that we consider will be finite.

1. INTRODUCTION

In this section, we shall introduce the many definitions and constructions that are required to study dimension. The 16 posets in the first figure are meant to provide the reader with some realistic examples in order to better appreciate the definitions and constructions we give. We want the reader to get a feeling for the main concepts before we get involved in difficult dimension calculations. No subsequent section requires any background except this introduction, and possibly, Section 3.

If a poset is denoted by P, then its underlying set will usually also be denoted by P, and its order relation by $\leq$ or $\leq_P$. An extension of P means an extension of $\leq_P$. If P is a subposet of Q, then $\dim P \leq \dim Q$ since any linear extension of Q restricts to a linear extension of P. T. Hiraguchi [1951] showed that the dimension increases by at most one when a single point is added. Equivalently, the "one-point removal theorem" is valid; that is, removing one point from a poset decreases its dimension by at most one. We shall consider this and other removal theorems in Sections 5 and 6.

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The converse of a binary relation R is denoted by R^d. Clearly, a poset P and its dual P^d have the same dimension. A poset is *d-irreducible* if it has dimension $d \geq 2$ and the removal of any element lowers its dimension. An *irreducible* poset is d-irreducible for some $d \geq 2$. By the compactness property, every irreducible poset is finite. By the one-point removal theorem, a poset is d-irreducible when $\dim P \geq d$ and $\dim(P - \{x\}) < d$ for every $x \in P$. The only 2-irreducible poset is the 2-element antichain (two incomparable elements). Figure 1 shows all 3-irreducible posets with at most 8 elements (up to isomorphism and duality). (We are using the notation of Kelly [1977].) Two of these posets have 6 elements, three have 8 elements, and the remainder have 7 elements. In Section 3, we show how to check whether a poset has dimension at most two.

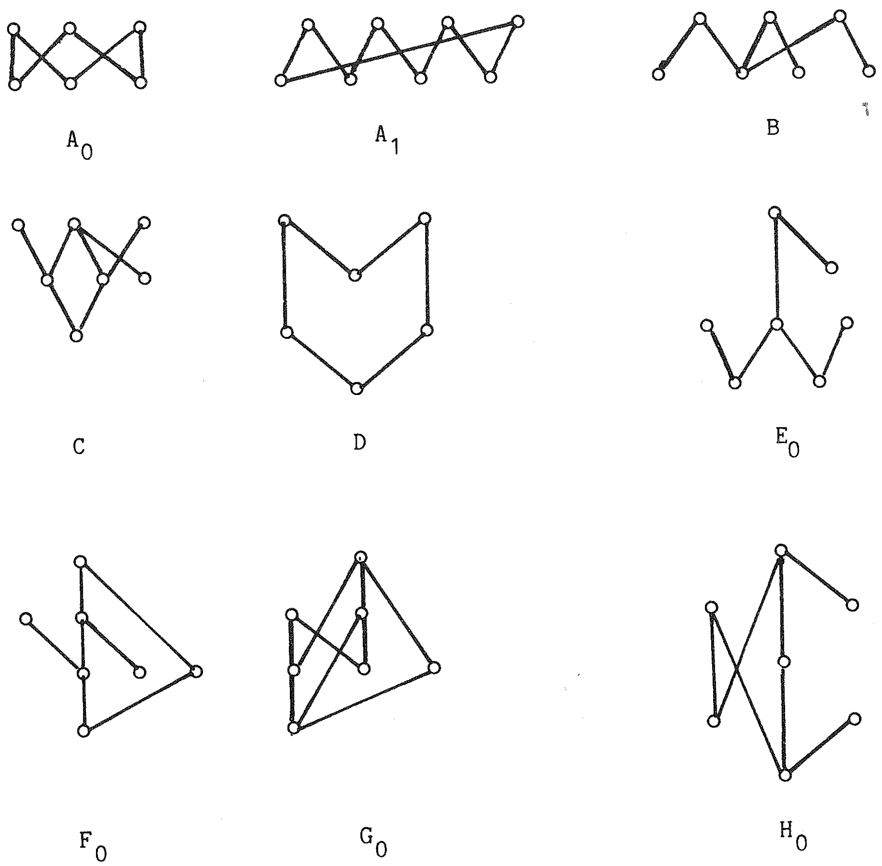

FIGURE 1. The 3-irreducible posets with at most 8 elements.

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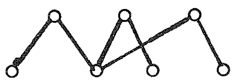

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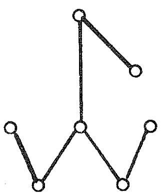
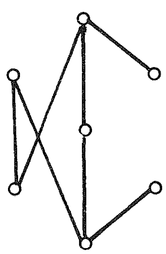


E_0



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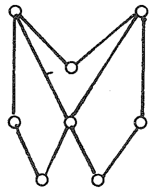
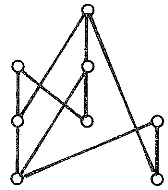


I_0



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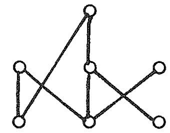
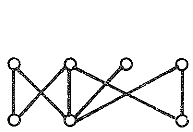
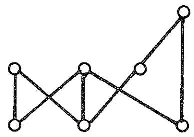


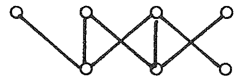
CX_1



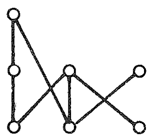
CX_2



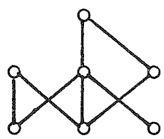
CX_3



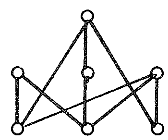
EX_1



EX_2



FX_1



FX_2

FIGURE 1. (concluded)

By Figure 1, any poset of dimension 3 must have at least 6 elements. In fact, this is a special case of the following inequality of Hiraguchi [1951], valid for $|P| \ge 4$:

$$\dim P \le |P|/2.$$

This result is proved in Section 5.

The posets A_0 and A_1 of Figure 1 are the two smallest members of an infinite family A_n $(n \ge 0)$ of 3-irreducible posets. These posets, called *crowns*, appeared in Harzheim [1970] and Baker, Fishburn and Roberts [1971]. D. Kelly and I.

Rival [1975b] discovered infinitely many more 3-irreducible posets. They presented the four infinite families E_n, F_n, G_n, H_n ($n \geq 0$), whose smallest members are shown in Figure 1. The list of all 3-irreducible posets was completed independently by D. Kelly [1977], and W.T. Trotter and J.I. Moore [1976a]. Up to duality, this list consists of the seven infinite families A_n, E_n, F_n, G_n, H_n, I_n, J_n ($n \geq 0$), whose smallest members are shown in Figure 1, together with the remaining posets of that figure. We shall return to the topic of irreducible posets in Section 7.

The following alternative definition is often credited to O. Ore [1962], but appeared earlier in Hiraguchi [1955] : The dimension of a poset P is the minimum size of a family $(C_i \mid i \in I)$ of chains such that P is isomorphic to a subposet of the direct product $\Pi(C_i \mid i \in I)$. If $(C_i \mid i \in I)$ is a family of linear extensions realizing P, then the diagonal mapping ϕ embeds P into the direct product of these chains ($\phi(x)$ is the constant function with value x for each x in P). To complete the proof of the equivalence of the two definitions, let P be the direct product of the family $(C_i \mid i \in I)$ of chains. For $i \in I$, we define the extension E_i of P by $\langle x, y\rangle \in E_i$ iff $\pi_i(x) < \pi_i(y)$, where π_i denotes the i-th projection function. If E_i' is a linear extension of E_i for each $i \in I$, then $(E_i' \mid i \in I)$ realizes P. H. Komm [1948] proved that $\dim \mathfrak{2}^{\mathfrak{m}} = \mathfrak{m}$, where $\mathfrak{2}$ denotes the 2-element chain and $\mathfrak{m}$ denotes a possibly infinite cardinal. It follows that the product of $\mathfrak{m}$ nontrivial chains has dimension $\mathfrak{m}$.

The Ore definition of dimension is often appropriate when no explicit calculation of dimension is required. When the dimension must be calculated, the original Dushnik-Miller definition is used. We shall describe a simple technique which reduces the "bookkeeping" involved in calculating the dimension of a finite poset.

A (linear) extension of a subposet of P will be called a *partial (linear) extension* of P. $\mathrm{Tr}(R)$ denotes the transitive closure of a binary relation R. For any partial extension E, $\mathrm{Tr}(\leq_P \cup E)$ is an extension of P. For a poset P, $\mathcal{I}(P) = \{\langle a,b\rangle \mid a \| b$ in $P\}$, the set of *incomparable pairs*. Any $u \in \mathcal{I}(P)$ can be considered as a partial linear extension of P.

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For $u, v \in \mathcal{I}(P)$, u *forces* v (in symbols, $\langle u,v\rangle \in F$) if $v \in \mathrm{Tr}(\leq_P \cup \{u\})$. This concept was introduced by I. Rabinovitch and I. Rival [1979], who observed that F is an order relation on $\mathcal{I}(P)$. The incomparable pair $\langle a,b\rangle$ is called a *critical pair* if $x < b$ implies $x < a$, and $x > a$ implies $x > b$. A pair $\langle a,b\rangle \in P^2$ is a *max-min pair* if a is maximal and b is minimal. Observe that every incomparable max-min pair is critical. For a poset P, Crit(P) denotes the set of critical pairs. Crit(P) is the set of minimal elements of $\langle \mathcal{I}(P);F\rangle$. We denote the set of maximal elements of $\langle \mathcal{I}(P);F\rangle$ by $\mathfrak{n}(P)$; these pairs have been called *unforced, nonforcing* and *nonforced*. We shall use the last term. Moreover, $\mathfrak{n}(P)$ is the converse of Crit(P). Thus, critical pairs are also called "reversed nonforced pairs". For any finite poset, every incomparable pair is forced by a critical pair.

For $S \subseteq \mathcal{I}(P)$, an *alternating cycle of length* n for S is a sequence of the form $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ with $\langle x_i, y_i\rangle \in S$, $1 \leq i \leq n$, and $y_i \leq x_{i+1}$ in P (with subscripts taken modulo n). S is *cycle-free* if it has no alternating cycle. The above cycle is *minimal* if $y_i \leq x_j$ implies $j = i+1 \pmod{n}$. If S is not cycle-free, then there is a minimal alternating cycle for S. If the above alternating cycle is minimal, then both $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ are n-element antichains. For example, $x_1 \leq x_i$ would imply that $y_n \leq x_i$. Thus, the length of a minimal alternating cycle for $\mathcal{I}(P)$ cannot exceed the width of the poset P.

We say that a family $(E_i \mid i \in I)$ of partial extensions *realizes* P (or $\leq_P$) when $(C_i \mid i \in I)$ realizes $\leq_P$ for any choice of linear extensions C_i of P that extend E_i ($i \in I$). The dimension of P is the minimum (nonzero) size of such a realizer. Let P be a poset in which every incomparable pair is forced by a critical pair. If P is not a chain, then dim P is specified by each of the following three definitions:

(a) The least number of partial extensions of P whose union contains Crit(P).

(b) The least number of partial linear extensions of P whose union contains Crit(P).

(c) The least number of cycle-free sets covering Crit(P).

The set $\mathrm{Crit}(P)$ is the smallest subset of $\mathcal{I}(P)$ that could be used in the above definitions; thus, $\mathrm{Crit}(P)$ is a "critical" set in dimension calculations. When using definition (a) or (b), we often write a critical pair $\langle a,b\rangle$ as "$a < b$" and call it a *critical inequality*.

Henceforth, any poset is understood to be finite unless the contrary is explicitly stated. P will always denote a poset. A realizer is *irredundant* if no proper subset is a realizer. The dimension is the smallest size of an irredundant realizer consisting of linear extensions, while the *rank* is the largest. Note that realizers used for rank must consist of linear extensions, in contrast to the case for dimension. For example, the realizer consisting of $\mathrm{Crit}(P)$, considered as partial linear extensions, is not permitted. Since any proper extension of P satisfies some nonforced pair, I. Rabinovitch and I. Rival [1979] observed that $\mathrm{rank}(P) \le |\mathfrak{n}(P)|$. S.B. Maurer, I. Rabinovitch and W.T. Trotter ([1980a], [1980b], [1980c]) describe how to determine the rank from $\mathfrak{n}(P)$, considered as a directed graph. Since $\mathfrak{n}(P)$ and $\mathrm{Crit}(P)$ are converses of each other, any extension satisfying all of $\mathfrak{n}(P)$ will fail every critical inequality. An extension satisfying all of $\mathfrak{n}(P)$ is called a *weak* extension. In Section 5, we show that every irreducible poset, except the 2-element antichain, has a weak extension.

We shall define a subposet $\underset{\sim}{P}(P)$, the set of *irreducible* elements of P, that has the same dimension as P. (The term "irreducible" has a completely different meaning when applied to elements than when applied to posets.) $\underset{\sim}{J}(P)$ is the subposet of *join-irreducible* elements ($a \in \underset{\sim}{J}(P)$ iff $a = \bigvee S$ implies $a \in S$). If P contains a least element 0, it is join-reducible since we allow $S = \emptyset$. $\underset{\sim}{M}(P)$, the subposet of meet-irreducible elements, is defined dually. We define

$$\underset{\sim}{P}(P) = \underset{\sim}{J}(P) \cup \underset{\sim}{M}(P),$$

a subposet of P.

If P is a lattice and $x \in P$, then $x \in \underset{\sim}{J}(P)$ iff x has a unique lower cover. In an arbitrary finite poset P, it is much harder to recognize the join-irreducible elements. This difficulty is overcome by constructing the finite lattice $L = \underset{\sim}{L}(P)$ which contains P as a subposet and satisfies $\underset{\sim}{J}(L) = \underset{\sim}{J}(P)$, $\underset{\sim}{M}(L) = \underset{\sim}{M}(P)$ and $\underset{\sim}{P}(L) = \underset{\sim}{P}(P)$. $\underset{\sim}{L}(P)$, the *completion* of P, is called the "completion by cuts" and defined in Birkhoff [1967]; it is also called the "MacNeille completion". The definition of $\underset{\sim}{L}(P)$ does not require P to be finite. Whenever a complete lattice contains a poset P, it contains the completion of P as a subposet. We shall not define the completion. Given