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THE INTERVAL NUMBER OF A COMPLETE MULTIPARTITE GRAPH

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The interval number of a graph G, denoted $i(G)$, is the least positive integer t for which G is the intersection graph of a family of sets each of which is the union of at most t closed intervals of the real line $\mathbb{R}$. Trotter and Harary showed that the interval number of the complete bipartite graph $K(m,n)$ is $\lceil (mn+1)/(m+n) \rceil$. Matthews showed that the interval number of the complete multipartite graph $K(n_1, n_2, \dots, n_p)$ was the same as the interval number of $K(n_1, n_2)$ when $n_1 = n_2 = \cdots = n_p$. Trotter and Hopkins showed that $i(K(n_1, n_2, \dots, n_p)) \le 1 + i(K(n_1, n_2))$ whenever $p \ge 2$ and $n_1 \ge n_2 \ge \cdots \ge n_p$. West showed that for each $n \ge 3$, there exists a constant c_n so that if $p \ge c_n$, $n_1 = n^2 - n - 1$, and $n_2 = n_3 = \cdots n_p = n$, then $i(K(n_1, n_2, \dots, n_p)) = 1 + i(K(n_1, n_2))$. In view of these results, it is natural to consider the problem of determining those pairs (n_1, n_2) with $n_1 \ge n_2$ so that $i(K(n_2, \dots, n_p)) = i(K(n_1, n_2))$ whenever $p \ge 2$ and $n_2 \ge n_3 \ge \cdots \ge n_p$. In this paper, we present constructions utilizing Eulerian circuits in directed graphs to show that the only exceptional pairs are $(n^2 - n - 1, n)$ for $n \ge 3$ and $(7,5)$.

1. Introduction

A graph G is an *interval graph* if it is the intersection graph of a family of closed intervals of the real line $\mathbb{R}$. Several authors have considered the following natural generalization of an interval graph. For an integer $t \ge 1$, a graph G is a *t-interval graph* if it is the intersection graph of a family of sets each of which is the union of at most t closed intervals of $\mathbb{R}$. More formally, G is a t-interval graph if there exists a function F which assigns to each vertex $x \in G$ a subset $F(x) \subseteq \mathbb{R}$ so that $F(x)$ is the union of at most t closed intervals of $\mathbb{R}$ and distinct vertices x and y are adjacent in G if and only if $F(x) \cap F(y) \ne \emptyset$. The function F is called a *t-interval representation* of G. The *interval number* of a graph G, denoted $i(G)$, is then defined as the least positive integer t for which G is a t-interval graph. Consequently, a graph G

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is a interval graph if and only if $i(G)=1$. An $i(G)$-interval representation is called an optimal interval representation.

In contrast to the situation with arbitrary graphs, complete multipartite graphs admit optimal representations with sufficient regularity to permit rapid computations and explicit constructions. To illustrate this point, we refer the reader to [4] where Trotter and Harary develop a formula for the interval number of the complete bipartite graph $K(m,n)$.

Theorem 1 [4]. *The interval number of the complete bipartite graph $K(m,n)$ is*

$$\lceil (mn+1)/(m+n) \rceil \quad \textit{for all } m,n \geq 1. \qquad \square$$

It follows from this formula that the interval number of the balanced bipartite graph $K(n,n)$ is $\lceil \frac{1}{2}(n+1) \rceil$, and thus the inequalities in the following two theorems are best possible.

Theorem 2 (Griggs and West [2]). *If the maximum degree of a vertex in a graph G is Δ, then $i(G) \leq \lceil \frac{1}{2}(\Delta+1) \rceil$.* $\square$

Theorem 3 (Griggs [1]). *If G is a graph on n vertices, then $i(G) \leq \lceil \frac{1}{4}(n+1) \rceil$.* $\square$

When researchers first began to investigate the interval number of the complete multipartite graph $K(n_1, n_2, \ldots, n_p)$ where $p \geq 2$ and $n_1 \geq n_2 \geq \cdots \geq n_p \geq 1$ it was discovered that the interval number of $K(n_1, n_2, \ldots, n_p)$ was closely related to the interval number of $K(n_1, n_2)$; some researchers conjectured that the numbers were always equal. In certain cases, the construction used by Trotter and Harary for $K(n_1, n_2)$ could be immediately extended to $K(n_1, n_2, \ldots, n_p)$; in other cases, the constructions could not be extended without modification, but it seemed reasonable to believe that suitable modifications could always be found. In support of the conjecture, Matthews produced new constructions to establish the following result.

Theorem 4 (Matthews [3]). *If $p \geq 2$ and $n_1 = n_2 = \cdots = n_p \geq 1$, then*

$$i(K(n_1, n_2, \ldots, n_p)) = i(K(n_1, n_2)). \qquad \square$$

However, D. West disproved the conjecture by establishing the following result.

Theorem 5 (West [6]). *For every integer $n \geq 3$, there exists a constant c_n so that if $p \geq c_n$, $n_1 = n^2 - n - 1$, and $n_2 = n_3 = \cdots = n_p = n$, then $i(K(n_1, n_2, \ldots, n_p)) = 1 + i(K(n_1, n_2))$.* $\square$

In view of Theorem 5, it might be conjectured that the interval number of a complete multipartite graph may exceed the interval number of the complete bipartite subgraph formed by the largest two parts by an arbitrarily large amount. However, Trotter and Hopkins extended Matthews' construction using concepts suggested by West's research and established the following upper bound.

Theorem 6 (Trotter and Hopkins [5]). *If $p \geq 2$ and $n_1 \geq n_2 \geq \cdots \geq n_p \geq 1$, then*

$$i(K(n_1, n_2, \ldots, n_p)) \leq 1 + i(K(n_1 n_2)). \qquad \square$$

In view of these results, it is natural to consider the following problem.

Problem. For what pairs of integers (n_1, n_2) with $n_1 \geq n_2$ is it true that whenever $p \geq 2$ and $n_2 \geq n_3 \geq \cdots \geq n_p$, we always have $i(K(n_1, n_2, \ldots, n_p)) = i(K(n_1, n_2))$?

The remainder of this paper is devoted to the solution of this problem. We will prove that the only exceptional values are $(n_1, n_2) = (n^2 - n - 1, n)$ for $n \geq 3$, as provided by Theorem 5, and $(n_1, n_2) = (7, 5)$. We devote the next two sections of the paper to developing the necessary notation and terminology. In Section 4, we develop some needed material on Eulerian circuits in directed graphs and present the proofs of the principal theorems in Sections 5 and 6.

2. Notation, terminology, and preliminaries

We use the symbols $\mathbb{N}$ and $\mathbb{R}$ to denote respectively the set of positive integers and the set of real numbers. For an integer $k \in \mathbb{N}$, we let $\mathbb{N}_k = \{n \in \mathbb{N}: n \leq k\}$. If I_1 and I_2 are closed intervals of $\mathbb{R}$, we write $I_1 < I_2$ when $x_1 < x_2$ for every $x_1 \in I_1$ and every $x_2 \in I_2$. Throughout the paper, we follow the convention of viewing a point as a closed interval, so when we write $I = [x, y]$, we only require $x \leq y$. We refer to a single point as a degenerate interval; larger intervals are called non-degenerate.

For integers m, n with $m \geq n \geq 1$, we let $K(m, n)$ denote the complete bipartite graph G whose vertex set is the union of two independent sets $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. We also let $K(m, n \cdot \infty)$ denote the infinite complete multipartite graph whose vertex set is the union of the infinite collection of independent sets $\{A, B_1, B_2, B_3, \ldots\}$ where $A = \{a_1, a_2, \ldots, a_m\}$ and $B_i = \{b_{i1}, b_{i2}, \ldots, b_{in}\}$ for each $i \geq 1$. For an integer $p \geq 1$, we let $K(m, n \cdot p)$ denote the complete $(p+1)$-partite subgraph of $K(m, n \cdot \infty)$ generated by $A \cup B_1 \cup B_2 \cup \cdots \cup B_p$. For obvious reasons, we do not distinguish between $K(m, n)$ and $K(m, n \cdot 1)$; however, when $p > 1$, the $(p+1)$-partite graph $K(m, n \cdot p)$ is different from the bipartite graph $K(m, np)$.

For simplicity we write $i(m, n)$ instead of $i(K(m, n))$ and $i(m, n \cdot p)$ instead of $i(K(m, n \cdot p))$. Although it is not immediately clear, the infinite graph $K(m, n \cdot \infty)$ has a finite interval number so we will also write $i(m, n \cdot \infty)$ instead of $i(K(m, n \cdot \infty))$.

The central problem of this paper can now be reformulated as follows.

Problem. *Determine those pairs of integers m, n with $m \geq n \geq 1$ for which $i(m, n \cdot \infty) = i(m, n)$.*

In order to clarify arguments and constructions which follow, we pause to develop some important conventions regarding diagrams of interval representations. In Fig. 1(a), we show a 2-interval representation of the graph G shown in Fig. 1(b). We follow the usual convention of using a horizontal line for the line $\mathbb{R}$ from which all intervals are chosen, but for visual clarity, we permit intervals to be displaced vertically. In the remainder of this section, and more formally in the next, we develop a notational description of representations of $K(m, n \cdot \infty)$ which we call a *frame*. We will define it gradually by building a correspondence between notation and interval representations.

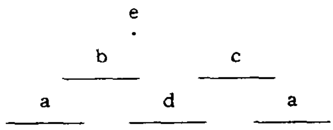


Fig. 1(a).

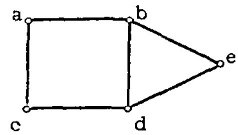


Fig. 1(b).

When presenting diagrams for t-interval representations of $K(m, n \cdot p)$, we agree to arrange the intervals in $p+1$ levels which we label from top to bottom Level 0, Level 1, Level 2, ..., Level p. At Level 0 appear those intervals corresponding to vertices in A. For each i with $1 \leq i \leq p$, we place at Level i those intervals which correspond to vertices in B_i. The use of levels simplifies the task of labelling the intervals in the diagram since we can label a Level 0 interval corresponding to a vertex a_α from A with the subscript α. Similarly when $1 \leq i \leq p$, we can label a Level i interval corresponding to a vertex b_{ij} from B_i with the single subscript j. With these conventions, we present in Figure 2 a diagram of a 3-interval representation of $K(4, 3 \cdot 2)$. Such diagrams correspond in a natural manner to interval representations of $(p+1)$-partite graphs and henceforth we will facilitate discussion by referring to such a diagram as an interval representation, which is a slight abuse of terminology.

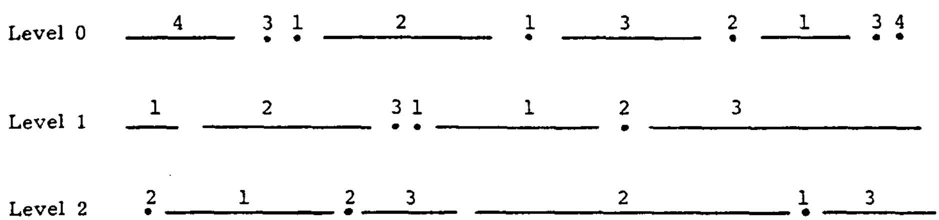


Fig. 2.

The reader may find it somewhat tedious to verify the claim that the diagram in Fig. 2 is a ctually a 3-interval representation of $K(4, 3 \cdot 2)$. Introducing the infinite graph $K(m, n \cdot \infty)$ will allow us to concentrate on more regular constructions. For example, consider the 3-interval representation of $K(4, 3 \cdot 2)$ shown in Fig. 3.

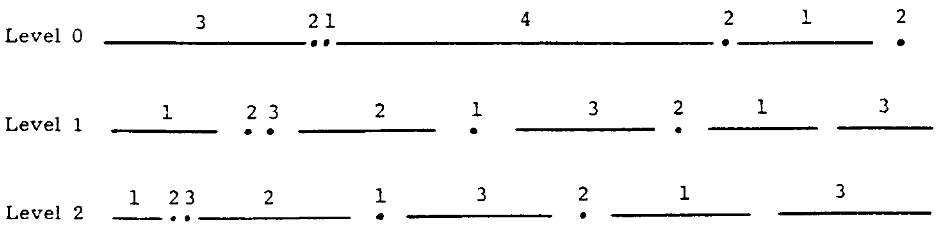


Fig. 3.

The obvious regularity present in Fig. 3 enables us to extend the construction to obtain a 3-interval representation of $K(4, 3\cdot p)$ for any $p \geq 1$. For example here is the diagram when $p=3$.

In fact, we may view the diagram in Fig. 4 as a 3-interval representation of $K(4, 3\cdot\infty)$ in which for the sake of clarity only the top few levels are shown. We encourage the reader to study Fig. 4 carefully in order to be convinced of the fact that the entire diagram is determined by specifying Level 0, Level 1 and the intersection pattern between these two levels.

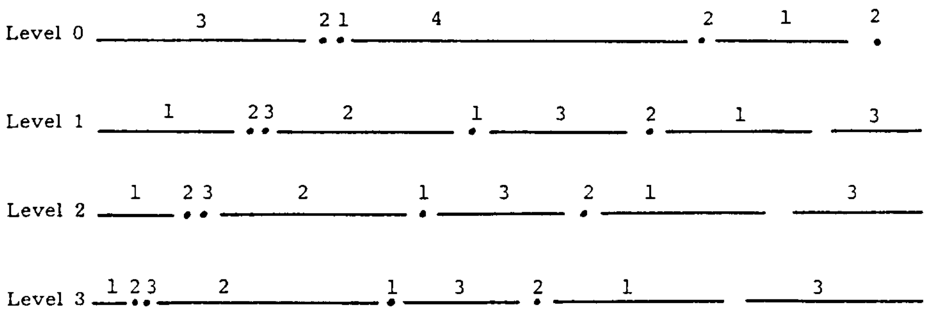


Fig. 4.

With these observations, it is not too difficult to develop a shorthand notation for regular constructions of the type illustrated in Fig. 4. We can begin by specifying the sequence σ of integers determined by reading the labels on the Level 1 intervals from left to right. In Fig. 4, the sequence σ is $(1,2,3,2,1,3,2,1,3)$. Certain positions in this sequence are distinguished by the fact that the corresponding intervals are nondegenerate. We can specify the distinguished positions by lowering the non-distinguished terms of σ to the level of a subscript. For Fig. 4, the pair σ and the set D of distinguished positions can then be specified by

$$1\;{}_{2}\;{}_{3}\;2\;{}_{1}\;3\;{}_{2}\;1\;3 \tag{1}$$

We can then specify the nondegenerate intervals at Level 0 by adding braces to indicate the location of their endpoints. Our example becomes

$$\overbrace{1\;{}_{2\;3}\;2}^{3}\;\overbrace{{}_{1}\;3\;{}_{2}}^{4}\;\overbrace{1}^{1}\;3 \tag{2}$$

Finally we can specify the location of the degenerate intervals at Level 0 by adding appropriate symbols at superscript level with respect to the symbols corresponding to nondegenerate intervals at Level 0. If there is no corresponding nondegenerate interval we will use the dummy symbol, $*$, as a place holder. For Fig. 4, we have

$$\overbrace{1\;{}_{2\;3}\;2}^{3^{2\;1}}\;\overbrace{{}_{1}\;3\;{}_{2}}^{4^{2}}\;\overbrace{1}^{1^{2}}\;3 \tag{3}$$

In what follows, we will refer to notation illustrated in (3) as a *frame*. To specify the values of m and n, we will also be viewed as a frame in which there are no degenerate symbols at Level 0.

To demonstrate the use of $*$, let us consider the 2-interval representation of $K(5, 2\cdot\infty)$ whose first two levels are specified in Fig. 5.

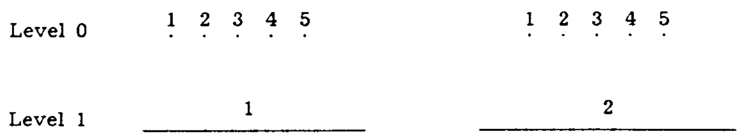


Fig. 5.

The corresponding (5,2)-frame is specified by

$$\overbrace{1\text{--}\qquad 2}^{1\;2\;3\;4\;5\;{}_{*}\;1\;2\;3\;4\;5}$$

Proceeding in the reverse direction, we can begin with any (m,n)-frame and associate with the frame a diagram of an interval representation of a subgraph of $K(m, n\cdot\infty)$. For example, with the following frame

$$2\;4\;\overbrace{1\;{}_{3\;4\;2}}^{5^{4\;1}}\;\overbrace{4\;{}_{1\;3}}^{3^{1\;2\;4}}\;2 \tag{4}$$

we associate a diagram for a 3-interval representation of a subgraph of $K(5, 4\cdot\infty)$. For clarity, we show in Fig. 6 only the top three levels of this diagram.

In the next section, we will develop more formal notation and terminology for (m,n)-frames. We will also develop criteria for determining when their diagrams are

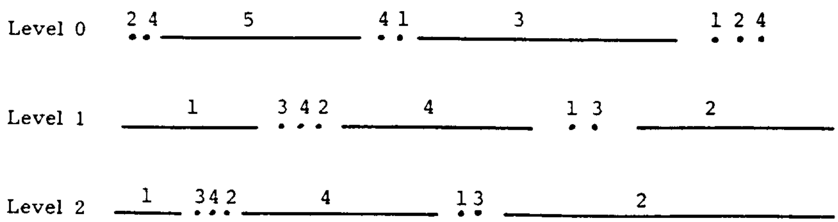


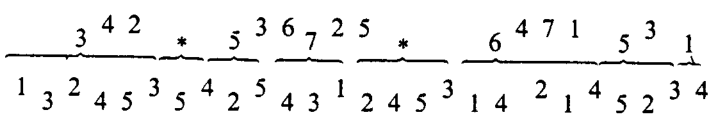
Fig. 6.

also interval representations of $K(m, n \cdot \infty)$. As a consequence, we will be able to present in Section 5 optimal t-interval representations in compact form. We will also establish the theoretical groundwork for handling the exceptional cases in Section 6.

3. Sequences and frames

Although frames provide us immediately with a clear and concise method for visualizing interval representations of subgraphs of $K(m, n)$, in order to use them in any theoretical discussion we must develop some more formal definitions and terminology. In this section we develop the kind of notation we will need in the arguments of Sections 4, 5, and 6.

A *sequence of length l* is a function σ from $\mathbb{N}_l$ to $\mathbb{N}$. We denote the length of a sequence σ by $l(\sigma)$ and the range of σ by $r(\sigma)$. For a subset $S \subseteq \mathbb{N}_l$, we let $\sigma(S) = \{\sigma(k): k \in S\}$ so that $r(\sigma) = \sigma(\mathbb{N}_l)$. For an integer $j \in r(\sigma)$, we define the *density of j in σ*, denoted $\delta_\sigma(j)$, as the number of times j appears as a term in σ. We define the *density of σ*, denoted $\delta(\sigma)$, by $\delta(\sigma) = \max\{\delta_\sigma(j): j \in r(\sigma)\}$. For the sequence σ of symbols at Level 1 in the (7,5)-frame shown in (5), we note that $l(\sigma) = 26$, $r(\sigma) = \mathbb{N}_5$, and $\delta(\sigma) = \delta_\sigma(4) = 7$.



(5)

A *sequence with distinguished positions* (a *DP-sequence* for short) is a pair (σ, D) where σ is a sequence and $D \subseteq \mathbb{N}_{l(\sigma)}$ with $\{1, l(\sigma)\} \subseteq D$. If $k \in D$, then we say that position k is *distinguished*; similarly, we say that $\sigma(k)$ occurs *in the distinguished position k*. The symbols at Level 1 in a frame determine a DP-sequence. For example (5), we note that the set D of distinguished positions is $\{1, 3, 6, 8, 10, 13, 17, 20, 22, 25, 26\}$. The requirement that $\{1, l(\sigma)\} \subseteq D$ corresponds to the fact that the intervals at the ends of a representation can be extended outward arbitrarily far. Note that in frames the sequences at Level 0 and Level 1 are both DP-sequences where the distinguished positions correspond to the nondegenerate intervals in the corresponding interval representations.

We say that a DP-sequence (σ, D) is *n-complete* when the range of σ is $\mathbb{N}_n$ and the frame with (σ, D) at both Level 1 and Level 0 corresponds to a $\delta(\sigma)$-interval representation of the complete multipartite graph $(K(n, n \cdot \infty)$. For example, $1_2 3_2 12$ is a 3-complete but $1_2 321$ is not. To see the latter, observe that in the interval representation produced by $1_2 321$, whenever $i_1 < i_2$, there is no interval corresponding to $b_{i_1,3}$ which intersects an interval corresponding to $b_{i_2,1}$ in its diagram.

More generally, examining which edges are represented by a DP-sequence leads to a precise characterization of n-complete DP-sequences. Consider the portion of the corresponding diagram lying on two distinct levels and between two consecutive nondegenerate intervals.

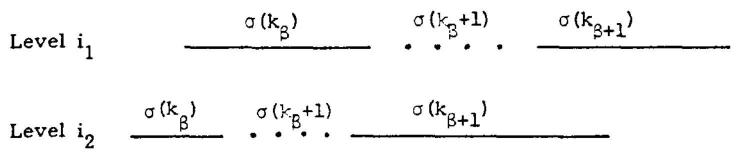


Fig. 7.

Examining the intersections in Fig. 7 yields the following formal characterization of n-complete DP-sequences.

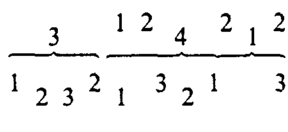
Lemma 1. *A DP-sequence (σ, D) is n-complete if and only if $r(\sigma) = \mathbb{N}_n$ and for every ordered pair (j_1, j_2) selected from $\mathbb{N}_n$, there exists an integer β* with $1 \le \beta < |D|$ *so that either*

(1) $j_1 = \sigma(k_\beta)$ *and* $j_2 \in \{\sigma(k): k_\beta \le k \le k_{\beta+1}\}$ *or*

(2) $j_2 = \sigma(k_{\beta+1})$ *and* $j_1 \in \{\sigma(k): k_\beta \le k \le k_{\beta+1}\}$. $\square$

To facilitate discussion, we may call such a pair of labels an *adjacency*.

We note that if (σ, D) is a n-complete, then we may conclude that the interval number of $K(n, n \cdot \infty)$ is at most $\delta(\sigma)$. However, in order to extend these observations to constructions for $K(m, n \cdot \infty)$ when $m > n$, we must allow for a new sequence of intervals at Level 0. We begin by defining notation to summarize the location of the nondegenerate intervals. As we proceed, we ask the reader to keep in mind an example like the following.



Let (σ, D) be a DP-sequence at Level 1 of some (m, n)-frame. Let E be the subset of D consisting of those distinguished positions for which the corresponding intervals contain an endpoint of a nondegenerate interval from Level 0 in the frame. In the frame notation these are the positions at the ends of the braces. We agree to use the symbol e to denote $|E|$ and when we write $E=\{h_1,h_2,\ldots,h_e\}$, it will be understood that $h_1<h_2<\cdots h_e$. There are $e-1$ nondegenerate intervals on Level 0, where the ith distinguished term corresponds to the brace from position h_i to position h_{i+1} in the frame. For $1\le\alpha\le m$, let $S_\alpha=\{i\in\mathbb{N}_{e-1}$: the ith distinguished term at Level 0 is α, i.e. the ith brace in the frame corresponds to a nondegenerate interval for $A_\alpha\}$. The collection $\mathscr{C}=\{S_\alpha\colon 1\le\alpha\le m\}$ is then an indexed collection of pairwise disjoint subsets of $\mathbb{N}_{e-1}$. If the symbol $*$ was used, then $\bigcup\mathscr{C}$ will not cover $\mathbb{N}_{e-1}$.

Occasionally, in the constructions which follow, all the intervals at Level 0 will be degenerate. In this case $E=\emptyset$ and $\mathscr{C}$ is a collection of empty sets.

The 4-tuple $(\sigma,D,E,\mathscr{C})$specifies everything about a frame except that it specifies no nondistinguished terms at Level 0. However, reducing our examination of frames to that class of frames will suffice, because we will next develop easy conditions on $(\sigma,D,E,\mathscr{C})$ so that adding nondistinguished terms at Level 0 in the frame it generates will yield a t-interval representation of $K(m,n\cdot\infty)$. Hence we can refer to such a frame or its interval diagram or its 4-tuple, $(\sigma,D,E,\mathscr{C})$, equivalently. Here is an example of a (9;7)-frame that works.

$$(\sigma,D,E,\mathscr{C})=\overbrace{1\;{}_{3}\,{}_{4}\;{}^{2}\;{}_{4}\,{}_{5}}^{2}\;\overbrace{{}^{3}\;{}_{5}\,{}_{6}}^{4}\;\overbrace{{}^{4}\;{}_{6}\,{}_{7}\;{}^{5}\;{}_{7}\,{}_{1}}^{3}\;{}^{6}\;\overbrace{{}_{1}\,{}_{2}}^{4}\;{}^{7}\;\overbrace{{}_{2}\,{}_{3}}^{5}\;\overbrace{1}^{6}\,\overbrace{5}^{7}\,\overbrace{2}^{8}\,\overbrace{6}^{4}\,\overbrace{3}^{9}\,\overbrace{7}^{8}\,\overbrace{4}^{1}\,1 \tag{6}$$

For this frame, $E=\{1,7,10,16,19,22,23,24,25,26,27,28,29\}$, $S_4=\{2,4,9\}$ and $S_8=\{8,11\}$. All other S_α's are singletons.

If we consider the interval representation of a subgraph of $K(m,n\cdot\infty)$ associated with an (m,n)-frame, we realize that for each level i, a nondegenerate interval for a_α overlaps an interval for b_{ij} for precisely those j's belonging to a set of the form $\{\sigma(h_\gamma),\sigma(h_\gamma+1),\ldots,\sigma(h_{\sigma+1})\}$ for some $\gamma\in S_\alpha$. Thus we are motivated to define, for each α, the overlap set of α relative to $\mathscr{C}$, denoted $\theta_{\mathscr{C}}(\alpha)$, by $\theta_{\mathscr{C}}(\alpha)=\{j\in\mathbb{N}_n\colon\sigma(k)=j$ for some $h_\gamma\le k\le h_{\gamma+1}$ with $\gamma\in S_\alpha\}$. When the context is clear, we may omit the subscript and write $\theta(\alpha)$ for $\theta_{\mathscr{C}}(\alpha)$.

We now turn our attention to the degenerate intervals at Level 0. As mentioned above, we will be less concerned about the precise location of these intervals. To illustrate this point, consider the following (5,4)-frame

$$\overbrace{1\;{}_{3}\;{}^{2}}^{3}\;\overbrace{{}_{4}\;{}^{3}\;{}_{1}}^{5}\;\overbrace{{}^{4}\;{}_{2}}^{2}\;\overbrace{1}^{4}\,\overbrace{3}^{1}\,\overbrace{2}^{5}\,4$$

It is easy to see that this frame can be suitably augmented to provide a representation for $K(5,4\cdot\infty)$ by adding degenerate intervals at Level 0. Examples include

$$\overbrace{1\;{}_{3}\;{}^{2}}^{3}\;{}^{2}\;\overbrace{{}_{4}\;{}^{3}\;{}_{1}}^{5}\;{}^{4}\;\overbrace{{}^{4}\;{}_{2}}^{{}^{1}2}\;{}^{1}\;\overbrace{1}^{4}\;\overbrace{3}^{1}\;{}^{4}\;\overbrace{2}^{5}\;4 \quad\text{and}\quad {}^{1}\;\overbrace{1\;{}_{3}\;{}^{2}}^{3}\;\overbrace{{}_{4}\;{}^{3}\;{}_{1}}^{5}\;{}^{1}\;\overbrace{{}^{4}\;{}_{2}}^{2}\;\overbrace{1}^{4}\;{}^{2}\;\overbrace{3}^{1}\;{}^{4}\;\overbrace{2}^{5}\;{}^{4}\;4$$

If $(\sigma, D, E, \mathscr{C})$ specifies an (m,n)-frame which can be augmented to provide a diagram for $K(m, n\cdot\infty)$ by adding degenerate intervals at Level 0, then we say $(\sigma, D, E, \mathscr{C})$ is *(m,n)-complete*. Roughly speaking all that is required is that each label in $\mathbb{N}_n$ appear at least once in a distinguished position of σ that is in E. For example, the left frame below is (4, 3)-complete but the right frame is not since no degenerate intervals for a_2 or a_3 can be inserted which overlap intervals corresponding to b_{i1} for each i.

```
   3   4
 1   2   3 4        1   2 3   4
----------------   ---------------
1 2 3 1 2  1 3     2 1 3 2 3 1 2
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The above discussion can be formalized into necessary and sufficient conditions for an (m,n)-frame to be (m,n)-complete.

Lemma 2. *An (m,n)-frame specified by $(\sigma, D, E, \mathscr{C})$ where $E=\{h_1, h_2, \ldots, h_e\}$ is (m,n)-complete if and only if the following two conditions hold.*

(1) *(σ, D) is n-complete.*

(2) *If $D_1=\{d_i \in D: d_i < h_1\}$, $D_2=\{d_i \in D: d_i \geq h_e\}$, and $N=\{i \in \mathbb{N}_n: i \in \theta(\alpha)$ for each $1 \leq \alpha \leq m\}$, then $N \cup \sigma(E) \cup \sigma(D_1) \cup \sigma(D_2) = \mathbb{N}_n$.*

Proof. The necessity and consequences of the first condition are obvious. The second condition simply prevents all intervals at level i corresponding to b_{ij} from being totally in the interior of nondegenerate intervals from Level 0, if in fact intersections with such an interval are needed to complete the representation. $\square$

Although there may be many different ways to add these degenerate intervals, the important observation to make is that for each α, we can easily determine the minimum number of degenerate intervals corresponding to a_α which must be added. When the (m,n)-frame is specified by $(\sigma, D, E, \mathscr{C})$ each α requires $n-|\theta_{\mathscr{C}}(\alpha)|$ degenerate intervals. Therefore we define *the density of α relative to $\mathscr{C}$*, denoted $\delta_{\mathscr{C}}(\alpha)$, by $\delta_{\mathscr{C}}(\alpha)=n-|\theta_{\mathscr{C}}(\alpha)|+|S_\alpha|$. We then define the density of $\mathscr{C}$, denoted $\delta(\mathscr{C})$, by $\delta(\mathscr{C})=\max\{\delta_{\mathscr{C}}(\alpha): 1\leq\alpha\leq m\}$. Thus in the (9, 7)-frame in example (6),

$$\theta_{\mathscr{C}}(2)=\{1,2,3,4,5\} \quad \text{so} \quad \delta_{\mathscr{C}}(2)=3;$$

$$\theta_{\mathscr{C}}(4)=\{1,2,3,4,5,6,7\} \quad \text{so} \quad \delta_{\mathscr{C}}(4)=|S_4|=3;$$

$$\theta_{\mathscr{C}}(6)=\{1,5\} \quad \text{so} \quad \delta_{\mathscr{C}}(6)=6; \quad \text{and} \quad \delta(\mathscr{C})=6.$$

Note that $\delta(\mathscr{C})$ gives the number of intervals 'needed' for vertices in a, and $\delta(\sigma)$ gives the number needed for vertices in B_i. Thus we have

Lemma 3. *If $(\sigma, D, E, \mathscr{C})$ determines an (m,n)-complete frame with $\delta=\max\{\delta(\sigma), \delta(\mathscr{C})\}$, then the interval representation associated with $(\sigma, D, E, \mathscr{C})$ is a δ-interval representation of $K(m, n\cdot\infty)$ and therefore $i(m, n\cdot\infty)\leq\delta$.* $\square$

To build complicated constructions, we need a way of putting DP-sequences together. Suppose σ_1 and σ_2 are sequences of length l_1 and l_2 respectively, and $\sigma_1(l_1)=\sigma_2(1)$. Then we define the *splice* of σ_1 and σ_2, denoted $\sigma_1 \oplus \sigma_2$, as the sequence σ of length l_1+l_2-1 for which $\sigma(k)=\sigma_1(k)$ for $1 \le k \le l_1$ and $\sigma(k)=\sigma_2(k-l_1+1)$ for $l_1 \le k \le l_1+l_2-1$. If (σ_1, D_1) and (σ_2, D_2) are DP-sequences and $\sigma_1(l_1)=\sigma_2(1)$, then we define the *splice* of (σ_1, D_1) and (σ_2, D_2), denoted $(\sigma_1, D_1)\oplus(\sigma_2, D_2)$, as the DP-sequence (σ, D) where $\sigma=\sigma_1\oplus\sigma_2$ and $D=D_1 \cup \{k\colon k-l_1+1 \in D_2\}$.

The notion of a splice is useful for specifying a DP-sequence which is to long to be written on a single line. In this case, the DP-sequence can be broken at a distinguished position. For example,

$$(\sigma, D) = {}^{1}\ 3\ 4\ {}^{2}\ 4\ 5\ {}^{3}\ 5\ 6\ {}^{4}\ 6\ 7\ {}^{5}\ 7\ 1\ {}^{6}\ 1\ 2\ {}^{7}\ 2\ 3\ {}^{1}$$

$$\oplus 1\ 5\ 2\ 6\ 3\ 7\ 4\ 1$$

Similarly if an (m,n)-frame $(\sigma, D, E, \mathscr{C})$ is too long for a single line, it can be broken at a position from E, as in the $(9,7)$-frame

$$\overbrace{{}^{1}\ 3\ 4}^{1}\ \overbrace{{}^{2}\ 4\ 5}^{2}\ \overbrace{{}^{3}\ 5\ 6}^{3}\ \overbrace{{}^{4}\ 6\ 7}^{4}\ \overbrace{{}^{5}\ 7\ 1}^{5}\ \overbrace{{}^{6}\ 1\ 2}^{6}\ \overbrace{{}^{7}\ 2\ 3}^{7}\ {}^{1}$$

$$\oplus\ \overbrace{1\ 5\ 2\ 6}^{8}\ \overbrace{3\ 7\ 4}^{9}\ 1$$

4. Eulerian circuits in directed graphs

In this section, we develop some specialized results concerning Eulerian circuits in directed graphs. These will be used to construct n-complete DP-sequences with various densities. The basic concepts in this section were introduced in [5], but we will require additional notation for the results in this paper. For an integer $n \le 3$, let $\mathbf{T}(n)$ denote the complete doubly directed graph on the vertex set $\{1,2,3,\ldots,n\}$. Its edge set is $\{(j_1, j_2)\colon 1 \le j_1, j_2 \le n, j_1 \ne j_2\}$. For integers n, s with $n \le 3$ and $1 \le s \le \lfloor \frac{1}{2}(n-1) \rfloor$, we define $\mathbf{T}(n,s)$ as the spanning subgraph of $\mathbf{T}(n)$ whose edge set[1] is $\{(j_1, j_2)\colon s+1 \le j_2 - j_1 \le n-s \text{ (cyclically)}\}$. $\mathbf{T}(n,s)$ is a regular graph in which vertex has outdegree and indegree $n-2s$.

For a subgraph $T \subseteq \mathbf{T}(n)$, let $E(T)$ denote the edge set of T. A sequence σ of length $|E(T)|$ is called an *Eulerian circuit of* T when $\sigma(1)=\sigma(|E(T)|)$ and for every edge $(j_1, j_2) \in E(T)$, j_1 is followed by j_2 exactly once in σ.

[1] Throughout the remainder of the paper, we will use the set $\{1,2,\ldots,n\}$ in cyclic fashion, i.e., $n+1=1$, $n+2=2$, etc.

Let $n \geq 3$ and $1 \leq s \leq \lfloor \frac{1}{2}(n-1) \rfloor$. Then we denote by $\sigma_{n,s}$ the sequence $(1, s+2, 2, s+3, 3, s+4, \ldots, n-1, s, n, s+1, 1)$. Note that $\sigma_{n,s}$ has length $2n+1$ when $1 \leq s < \frac{1}{2}(n-1)$, but has length $n+1$ when $s = \frac{1}{2}(n-1)$. For example,

$$\sigma_{8,2} = (1, 4, 2, 5, 3, 6, 4, 7, 5, 8, 6, 1, 7, 2, 8, 3, 1),$$

$$\sigma_{8,3} = (1, 5, 2, 6, 3, 7, 4, 8, 5, 1, 6, 2, 7, 3, 8, 4, 1),$$

$$\sigma_{7,3} = (1, 5, 2, 6, 3, 7, 4, 1).$$

When $s = \lfloor \frac{1}{2}(n-1) \rfloor$, $\sigma_{n,s}$ is an Eulerian circuit of $\mathbf{T}(n,s)$, and when $s < \lfloor \frac{1}{2}(n-1) \rfloor$, we observe that $\sigma_{n,s}$ is an Eulerian circuit of the spanning subgraph of $\mathbf{T}(n)$ containing all edges which belong to $\mathbf{T}(n,s)$ but not $\mathbf{T}(n,s+1)$.

It follows that if $1 \leq s \leq \lfloor \frac{1}{2}(n-1) \rfloor$, then the sequence $t_{n,s}$ defined by $t_{n,s} = \sigma_{n,s} \oplus \sigma_{n,s+1} \oplus \sigma_{n,s+2} \oplus \cdots \oplus_{n,\lfloor (n-1)/2 \rfloor}$ is an Eulerian circuit of $\mathbf{T}(n,s)$ of length $n(n-2s)+1$. In what follows we must also be concerned about sets of consecutive terms of $t_{n,s}$.

Lemma 4. *Let $n \geq 3$ and $1 \leq s \leq \lfloor \frac{1}{2}(n-1) \rfloor$. Then*

$$t_{n,s} = \sigma_{n,s} \oplus \sigma_{n,s+1} \oplus \sigma_{n,s+2} \oplus \cdots \oplus \sigma_{n,\lfloor (n-1)/2 \rfloor}$$

is an Eulerian circuit of $\mathbf{T}(n,s)$. Furthermore, any $2s+1$ consecutive terms of $t_{n,s}$ are distinct.

Proof. We have already noted that $t_{n,s}$ is an Eulerian circuit of $\mathbf{T}(n,s)$. We now show that any $2s+1$ consecutive terms of $t_{n,s}$ are distinct. We have two cases here depending on whether these consecutive terms of $t_{n,s}$ come from a single $\sigma_{n,s}$ or overlap $\sigma_{n,s'}$ and $\sigma_{n,s'+1}$. First suppose the former; we have a set $S = \{t_{n,s}(k): k_0 \leq k \leq k_0 + 2s\}$ of $2s+1$ consecutive terms which belong to a sequence $\sigma_{n,s'}$ where $s \leq s' \leq \lfloor \frac{1}{2}(n-1) \rfloor$. Let $t_{n,s}(k_0) = j_0$. Then it follows that we have $S = S_1 \cup S_2$ where either

$$S_1 = \{j_0 + j : 0 \leq j \leq s\} \quad \text{and} \quad S_2 = \{j_0 + s' + 1 + j : 0 \leq j \leq s-1\},$$

or

$$S_1 = \{j_0 + j : 0 \leq j \leq s\} \quad \text{and} \quad S_2 = \{j_0 - s' + j : 0 \leq j \leq s-1\}.$$

Here the sets S_1 and S_2 consist of $s+1$ and s consecutive integers respectively (in the cyclic sense) of $\{1, 2, \ldots, n\}$. Furthermore, the inequalities $s < s'+1$ and $s + s' < n$ imply that $S_1 \cap S_2 = \emptyset$. Thus $|S| = |S_1| + |S_2| = 2s+1$.

It remains only to consider a set S in which there exist integers s', k_0 with $s \leq s' < \lfloor \frac{1}{2}(n-1) \rfloor$ and $1 \leq k_0 \leq 2s+1$ so that S consists of the last k_0 terms of $\sigma_{n,s'}$ and the first $2s+1-k_0$ terms of $\sigma_{n,s'+1}$. Again, let j_0 be the first term of S. Then $S = S_1 \cup S_2$ where either

$$S_1 = \{j_0 + j : 0 \leq j \leq s\} \quad \text{and}$$

$$S_2 = \{j_0 + s' + j + 1 : 0 \leq j \leq s, j_0 + s' + j + 1 \neq s' + 2\},$$

or

$$S_1=\{j_0+j: 0\le j\le s+1, j_0+j\ne s'+2\} \quad \text{and}$$

$$S_2=\{j_0-s'+j: 0\le j\le s-1\}.$$

As before, it is easy to see that we must have $|S_1|=s+1$ and $|S_2|=s$. Furthermore, the inequalities $s<s'+1$ and $s+s'+1<n$ imply that $S_1\cap S_2=\emptyset$ and thus $|S|=|S_1|=2s+1$. $\square$

For example,

$$t_{9,2}=(1,4,2,5,3,6,4,7,5,8,6,9,7,1,8,2,9,3,1,5,2,6,3,7,4,8,5,9,6,1,$$
$$7,2,8,3,9,4,1,6,2,7,3,8,4,9,5,1)$$

is an Eulerian circuit of $T(9,2)$ in which any five consecutive terms are distinct. In general, note that when $n\ge 3$, $1\le s\le \lfloor \frac{1}{2}(n-1)\rfloor$, and $\sigma=t_{n,s}$, we always have $\delta_\sigma(j)=n-2s$ for $1<j\le n$ and $\delta_\sigma(1)=\delta(\sigma)=n-2s+1$.

When $n\ge 3$ and $1\le s\le \lfloor \frac{1}{2}n\rfloor$ (note that this differs from the domain of $\sigma_{n,s}$), we define the sequence $\lambda_{n,s}$ of length $ns+1$ by the following rules. For $1\le j\le n+1$,

$$\lambda_{n,s}((j-1)s+1)=j \qquad \text{(cyclically)},$$

and for $1\le j_1\le n$, $1\le j_2\le s-1$,

$$\lambda_{n,s}((j_1-1)s+j_2+1)=j_1+j_2+1.$$

For example,

$$\lambda_{8,3}=(1,3,4,2,4,5,6,4,6,7,5,7,8,6,8,1,7,1,2,8,2,3,1).$$

Note that if $\sigma=\lambda_{n,s}$, then $\delta_\sigma(j)=s$ when $1<j\le n$ and $\delta_\sigma(1)=\delta(\sigma)=s+1$.

When $n\ge 3$ and $1\le s\le \lfloor \frac{1}{2}n\rfloor$, we turn $\lambda_{n,s}$ into a DP-sequence by setting $\Lambda_{n,s}=\{(j-1)s+1: 1\le j\le n+1\}$. For example, when $n=8$, $s=3$, $(\lambda_{8,3},\Lambda_{8,3})$ is

$${}^{1}\;3\;4\;{}^{2}\;4\;5\;{}^{3}\;5\;6\;{}^{4}\;6\;7\;{}^{5}\;7\;8\;{}^{6}\;8\;1\;{}^{7}\;1\;2\;{}^{8}\;2\;3\;{}^{1}$$

The following facts about $(\lambda_{n,s},\Lambda_{n,s})$ follow directly from its construction.

Lemma 5. *For $1\le s\le \lfloor \frac{1}{2}n\rfloor$, $(\lambda_{n,s},\Lambda_{n,s})$ is a DP-sequence with k in the kth distinguished position (cyclically). Furthermore, it contains all ordered pairs (j_1,j_2) with $1\le j_2-j_1\le s$ or $-1\ge j_2-j_1\ge -s$ as adjacencies (in the sense of Lemma 1).* $\square$

This lemma leads us to the usefulness of these Eulerian circuits. Note that for $1\le s\le \lfloor \frac{1}{2}n\rfloor$, the DP-sequence $(\lambda_{n,s},\Lambda_{n,s})$ if n-complete if and only if $s=\lfloor \frac{1}{2}n\rfloor$. In particular, note that the $(m,1)$-frame

$$\frac{*}{1}$$

is $(n,1)$-complete where $\sigma=\{1\}$, $D=\{1\}$, and $E=\emptyset$. Moreover $\delta(\mathscr{C})=1$, $\delta(\sigma)=1$ and thus $i(n,1\cdot\infty)=1$ for every $m\geq 1$.

Similarly the $(m,2)$-frame

$$\overset{*}{\overbrace{1\quad 2}}\overset{*}{\overbrace{\quad 1}}$$

is $(m,2)$-complete with $\sigma=\{1,2,1\}$, $D=\{1,2,3\}$, $E\neq\emptyset$ and $\delta(\mathscr{C})=\delta(\sigma)=2$, so $i(m,2\cdot\infty)=2$ for all $m\geq 2$.

When $n\geq 3$, Lemma 5 shows that the DP-sequence (σ,D) where $\sigma=\lambda_{n,\lfloor n/2\rfloor}$, $D=\Lambda_{n,\lfloor n/2\rfloor}$ is n-complete, with $\delta(\sigma)=1+\lfloor\frac{1}{2}n\rfloor\leq n$. We can then set $E=\emptyset$, so that $\mathscr{C}=\emptyset$, and then $(\sigma,D,E,\mathscr{C})$ determines an (m,n)-complete frame with $\delta(\mathscr{C})=n$ and $i(m,n\cdot\infty)\leq n$. We conclude that $i(m,n\cdot\infty)\leq n$ for all $n\geq 1$.

Here is a short proof of Theorem 6 using the notation of the last two sections. We include this argument since the same approach will be required in the proof of our principal theorem.

Theorem 6 [5]. *If $p\geq 2$ and $n_1\geq n_2\geq\cdots\geq n_p\geq 1$, then*

$$i(n_1,n_2,\ldots,n_p)\leq i(n_1,n_2)+1.$$

Proof. Let $m\geq n\geq 1$, $t=i(m,n)$ and $s=n-t$. We show that $i(m,n\cdot\infty)\leq t+1$. Since $i(m,n\cdot\infty)\leq n$ for all $n\geq 1$, the theorem holds for $n\leq 2$, so we may assume without loss of generality that $n\geq 3$. Thus $1\leq s\leq\lfloor\frac{1}{2}(n-1)\rfloor$. Then let

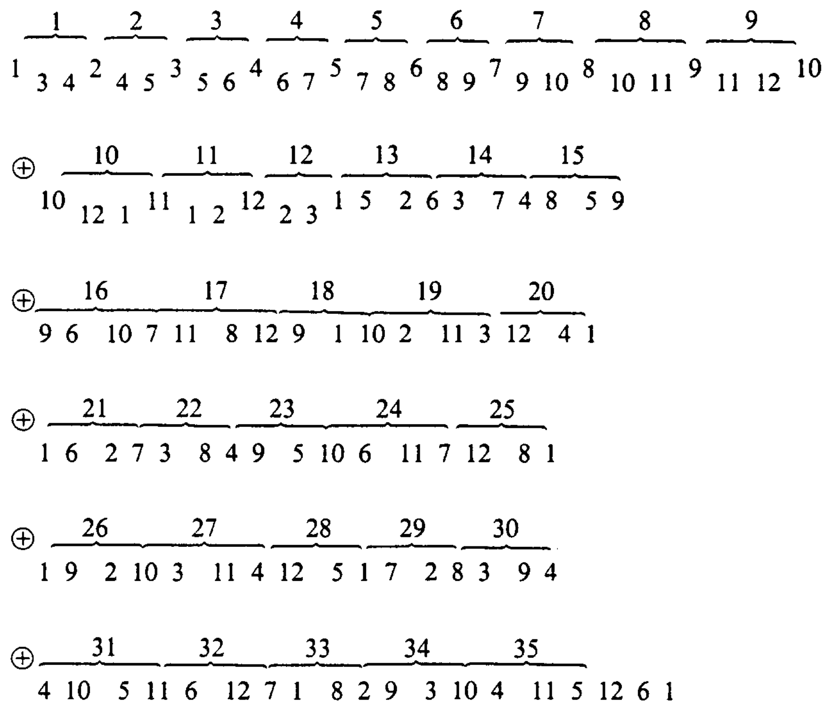
$$(*)\qquad \begin{aligned}&(\sigma,D)=(\lambda_{n,s},\Lambda_{n,s})\oplus(t_{n,s},\mathbb{N}_{n,n-2s+1}), \qquad E=\Lambda_{m,s},\\ &S_\alpha=\{\alpha\}\quad\text{for } \alpha=1,2,\ldots,m \quad\text{and}\quad \mathscr{C}=\{S_\alpha\colon 1\leq\alpha\leq m\}.\end{aligned}$$

We show first the (σ,D) is n-complete. We must show that all the adjacencies (j_1,j_2) with $j_i\in\mathbb{N}_n$ arise in (σ,D). Here we have two cases. If $j_1-j_2\leq s+1$ (cyclically), then the pair (j_1,j_2) appears as an adjacency in $(\lambda_{n,s},\Lambda_{n,s})$ as noted in Lemma 5. If on the other hand $s+1\leq j_1-j_2 n-s$ (cyclically), then somewhere in $t_{n,s}$ j_2 follows j_1 immediately. Since all positions in $t_{n,s}$ are distinguished this suffices. The conditions of Lemma 1 are satisfied for all pairs, and thus (σ,D) is n-complete.

As noted in Lemma 5, the first n distinguished positions in (σ,D) ae filled by $\{1,\ldots,n\}$. Since $\Lambda_{n,s}\subseteq\Lambda_{m,s}$ for $n\leq m$, we have $\sigma(E)=\mathbb{N}_n$. By Lemma 2, then $(\sigma,D,E,\mathscr{C})$ is (m,n)-complete. Furthermore, counting the appearances of each label yields $\delta_\sigma(j)=t$ for $1<j\leq n$, while $\delta_\sigma(1)=\delta(\sigma)=t+1$.

To complete the proof, it suffices to show that $\delta(\mathscr{C})=t$. First let $1\leq\alpha\leq n$. Then $\theta_{\mathscr{C}}(\alpha)=\{\alpha+j\colon 0\leq j\leq s\}$ so that $\delta_{\mathscr{C}}(\alpha)=n+|S_\alpha|-|\theta_{\mathscr{C}}(\alpha)|=n+1-(s+1)=n-s=t$. On the other hand, if $n>\alpha\leq m$, then $\theta_{\mathscr{C}}(\alpha)$ is a set of $s+1$ consecutive terms from the Euler circuit $t_{n,s}$ of $\mathbf{T}_{n,s}$. Since any $2s+1$ terms from $t_{n,s}$ are distinct, we conclude again that $\delta_{\mathscr{C}}(\alpha)=t$, completing the proof. $\square$

We illustrate the preceding theorem when $(m, n) = (35, 12)$. Degenerate intervals at Level 0 are omitted to avoid clutter. They can be placed above the first n distinguished positions in (σ, D).



5. Principal theorem

We begin this section with the following elementary result whose straightforward proof is omitted.

Lemma 6. *Let* $m \geq n \geq 1$. *If* $t = i(m,n) = \lceil (mn+1)/(m+n) \rceil$ *and* $s = n - t$, *then* $ms + 1 \leq nt$ *with equality holding if and only if* $(mn+1)/(m+n)$ *is an integer.* □

We say that the pair (m, n) is *tight* when $m \geq n \geq 1$ and $(mn+1)/(m+n)$ is an integer. For example $(13, 8)$ and $(9, 7)$ are tight but $(12, 6)$ is not. Note that $(m, 1)$ is tight for every $m \geq 1$ and that $(m, 2)$ is never tight when $m \geq 2$. It is straightforward to verify that the only tight pairs (m, n) with $3 \leq n \leq 8$ are: $(5,3), (11,4), (7,5), (19,5)$, $(29,6), (9,7), (17,7), (41,7), (13,8)$, and $(55,8)$.

Theorem 7. *Let* m, n *be integers with* $m \geq n \geq 1$. *If* $m \neq n^2 - n - 1$ *and* $(m,n) \neq (7,5)$, *then the interval number for the complete multipartite graph* $K(m, n \cdot \infty)$ *is equal to the interval number of the corresponding complete bipartite graph* $K(m, n)$.

Proof. The result is true when $n \geq 2$, as noted in Section 3, so we may assume $n \geq 3$. Now suppose $m \geq n^2 - n$. Then

$$n = i(n^2 - n, n) \leq i(m, n) \leq i(m, n \cdot \infty) \leq n,$$

so $i(m, n \cdot \infty) = i(m, n) = n$. We may therefore assume that $m < n^2 - n - 1$ and that $(m, n) \neq (7, 5)$. Throughout the remainder of the argument, we let $t = i(m, n) = \lceil (mn+1)/(m+n) \rceil$ and $s = n - t$. Note that $\lceil \frac{1}{2}(n+1) \rceil \leq t \leq n-1$ and $1 \leq s \leq \lfloor \frac{1}{2}(n-1) \rfloor$. Furthermore, note that $ms + 1 \leq nt$ with equality holding if and only if $t = (mn+1)/(m+n)$. We now proceed to show that $i(m, n \cdot \infty) \leq t$ by constructing a 4-tuple $(\sigma_0, D_0, E_0, \mathscr{C}_0)$ such that $(\sigma_0, D_0, E_0, \mathscr{C}_0)$ is (m, n)-complete, and $\delta(\sigma_0) = \delta(\mathscr{C}_0) = t$.

In all cases, this 4-tuple is obtained by modifying the 4-tuple $(\sigma, D, E, \mathscr{C})$ constructed in the proof of Theorem 6 in (*). The only thing that prevented that 4-tuple from being suitable here was $\delta_\sigma(1) = t + 1$. Therefore, we must modify $(\sigma, D, E, \mathscr{C})$ to save one interval for 1 in σ without losing any of the other completeness or density properties.

Case 1: (m, n) is not tight.

Begin with $(\sigma, D, E, \mathscr{C})$ defined as in (*), so that

$$(\sigma, D) = (\lambda_{n,s}, \Lambda_{n,s}) + (t_{n,s}, \mathbb{N}_{n(n-2s)+1}).$$

Consider the corresponding (m, n)-frame. Note that the last three terms of σ, all distinguished, are n, $\lceil \frac{1}{2}n \rceil$, 1. The last two distinguished terms of $(\lambda_{n,s}, \Lambda_{n,s})$ are n and 1. Hence, if we delete the last two terms of the sequence σ, we may lose the adjacencies $(n, \lceil \frac{1}{2}n \rceil)$ and $(\lceil \frac{1}{2}n \rceil$. However, we regain them by inserting $\lceil \frac{1}{2}n \rceil$ as a non-distinguished element between the last two distinguished elements of $(\lambda_{n,s}, \Lambda_{n,s})$. The resulting DP-sequence (σ_0, D_0) is still n-complete and $\delta(\sigma_0) = t$. The technical statement of the changes in the 4-tuple to produce $(\sigma_0, D_0, E_0, \mathscr{C}_0)$ from $(\sigma, D, E, \mathscr{C})$ can be read directly from the single change described in the frame. The proof that $(\sigma_0, D_0, E_0, \mathscr{C}_0)$ is (m, n)-complete and that $\delta(\mathscr{C}_0) = t$ follows through exactly as in Theorem 6, uneffected by the small change we made in the frame.

We illustrate this construction with the case (35, 12) and encourage the reader to compare it with the example at the end of Section 4.

$$ {}^{1}\overbrace{3\ 4}^{1}\,{}^{2}\overbrace{4\ 5}^{2}\,{}^{3}\overbrace{5\ 6}^{3}\,{}^{4}\overbrace{6\ 7}^{4}\,{}^{5}\overbrace{7\ 8}^{5}\,{}^{6}\overbrace{8\ 9}^{6}\,{}^{7}\overbrace{9\ 10}^{7}\,{}^{8}\overbrace{10\ 11}^{8}\,{}^{9}\overbrace{11\ 12}^{9}\,{}^{10} $$

$$ \oplus\ \overbrace{{}^{10}\ 12\ 1}^{10}\cdot\overbrace{{}^{11}\ 1\ 2}^{11}\ \overbrace{{}^{12}\ 2\ 3\ 6}^{12}\ {}^{1} $$

$$ \oplus\ \overbrace{1\ 5\ 2\ 6}^{13}\ \overbrace{3\ 7\ 4\ 8}^{14}\ \overbrace{5\ 9}^{15}\ \overbrace{6\ 10}^{16}\ \overbrace{7\ 11\ 8\ 12}^{17}\ \overbrace{9\ 1\ 19}^{18}\ \overbrace{2\ 11}^{19}\ \overbrace{3\ 12\ 4\ 1}^{20} $$

$$\oplus \quad 21 \quad 22 \quad 23 \quad 24 \quad 25 \quad 26 \quad 27 \quad 28$$
$$1\ 6\ 2\ 7\ 3\ 8\ 4\ 9\ 5\ 10\ 6\ 11\ 7\ 12\ 8\ 1\ 9\ 2\ 10\ 3\ 11\ 4\ 12\ 5\ 1$$

$$\oplus \quad 29 \quad 30 \quad 31 \quad 32 \quad 33 \quad 34 \quad 35$$
$$1\ 7\ 2\ 8\ 3\ 9\ 4\ 10\ 5\ 11\ 6\ 12\ 7\ 1\ 8\ 2\ 9\ 3\ 10\ 4\ 11\ 5\ 12$$

Unfortunately, this construction does not work when (m,n) is tight. Note that the original frame $(\sigma, D, E, \mathscr{C})$ has m brackets, extending over positions (h_i, h_i+s) which correspond to the m nondegenerate intervals used for Level 0. When the change is made, σ loses two positions at the end and one of the brackets in the middle becomes one position longer. We must show that σ is long enough to allow this. In σ_0, we have nt positions so we need $1+ms+1 \le nt$. This holds when (m,n) is not tight, i.e. $ms+1<nt$, but not when (m,n) is tight.

Case 2: (m,n) is tight.

In this case, we note that $s \ge 2$ and $t \le n-2$, for if $s=1$, then $t=n-1$ which requires $m=n^2-n-1$. In view of our preliminary remarks concerning tight pairs, we may also assume without loss of generality that $n \ge 7$.

As in Case 1, we begin with the sequence $(\sigma, D) = (\lambda_{n,s}, \Lambda_{n,s}) \oplus (t_{n,s}, \mathbb{N}_{n(n-2s)+1})$. Then let l denote the length of the sequence $\sigma_{n,s}$. Note that either $l=2n+1$ or $l=n+1$. Again we want to modify (σ, D) to decrease $\delta(1)$ by 1. However, now we must be more clever in defining (σ_0, D_0) to be able to extend it to an (m,n)-complete $(\sigma_0, D_0, E_0, \mathscr{C}_0)$. Technically, we define σ_0 by the following rule:

$$\sigma_0(k) = \begin{cases} \sigma(k) & \text{when } 1 \le k \le s-1, \\ s+2 & \text{when } k=s, \\ \sigma(k) & \text{when } s+1 \le k \le (n-2)s+1, \\ s & \text{when } k=(n-2)s+2, \\ \sigma(k) & \text{when } (n-2)s+3 \le k \le ns+1, \\ n & \text{when } k=ns+2, \\ s+1 & \text{when } k=ns+3, \\ \sigma(k-1) & \text{when } ns+4 \le k \le ns+l-3, \\ s+1 & \text{when } k=ns+l-2, \\ \sigma(k+1) & \text{when } ns+l-1 \le k \le nt. \end{cases}$$

Note that the definition of σ_0 requires $s \ge 2$ and $l \ge 7$, i.e. $n \ge 6$, if $l=n+1$. This observation explains in part why the pair $(7,5)$ is exceptional.

We have given the technical definition of σ_0 because more 'little changes' have been made than in Case 1. However, the argument can also be made pictorially. In effect, what we have done is to drop the 1 from position $(n-2)s+2$ in σ and rearrange a few other elements so that n-completeness will be maintained. We can illustrate the construction in general by the following diagram, which shows how (σ, D) is modified to obtain (σ_0, D_0). All the changes are indicated by arrows. Technically, we define

$$D_0 = \Lambda_{n,s} \cup \{k\colon ns+3 \le k \le nt, k \ne ns+l-2\}.$$

Switching with the fact that (σ, D) is n-complete, it is easy to verify that (σ_0, D_0) is n-complete by looking at Fig. 8. In switching from (σ, D) to (σ_0, D_0) a number of pairs of adjacencies are disturbed. However, a quick glance suffices to see that all these adjacencies are restored somewhere else in (σ_0, D_0), so Lemma 1 still applies and (σ_0, D_0) is n-complete. Note also that $\delta_{\mathscr{C}_0}(\alpha) = \delta(\sigma_0) = t$ for all α, $1 \le \alpha \le m$.

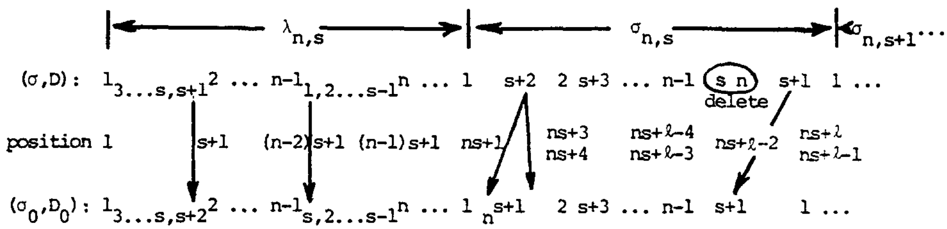


Fig. 8.

Now we must determine the set of positions E_0 to be used as endpoints of braces in the frame. Here we must divide the argument into three further cases. With $ms+1 = nt$ positions in σ_0, we will succeed easily if we can use m braces that each extend for s positions. This depends on whether the needed positions are distinguished, i.e. contained in D_0. The undistinguished element $s+1$ in position $ns = l-2$ is the only element that can cause a problem. It can cause a problem only if $ns+l-2 \equiv 1 \bmod s$, i.e. s divides $l-3$. If $l = n+1$, we would have $s = \frac{1}{2}(n-2)$ and s divides $n-2$, which can happen only if $s = 1$. So problems can occur only when $l = 2n+1$ and s divides $2n-2$.

Case a: $\Lambda_{m,s} \subsetneq D_0$, *i.e.* $s \nmid 2(n-1)$.

This is the 'main' subcase. Here we make all the brackets in the frame the same length s, and we label them consecutively from 1 to m. More technically we set $E_0 = \Lambda_{m,s}$, $S_\alpha = \{\alpha\}$ for $\alpha = 1, 2, \ldots, m$ and as usual $\mathscr{C}_0 = \{S_\alpha\colon 1 \le \alpha \le m\}$. As before the element of σ_0 indexed by the jth value in $\Lambda_{m,s}$ is j for $j = 1, \ldots, n$, so $\sigma(E_0) = \mathbb{N}_n$ and $(\sigma_0, D_0, E_0, \mathscr{C}_0)$ is (m,n)-complete. We need only show that $\delta(\mathscr{C}_0) = t$. It suffices to show that $|\theta_{\mathscr{C}_0}(\alpha)| = s+1$ for each $\alpha = 1, 2, \ldots, m$, i.e., each bracket overlaps $s+1$ distinct labels in σ_0.

As in the non-tight case, the overlap sets are only slightly modified from $\theta_{\mathscr{C}}(\alpha)$; we must show that the substitutions don't produce duplicate values in σ_0 within a single brace. Within the braces covering $\lambda_{n,s}$, the initial portion of σ, only two changes take place, in $\theta(1)$ and $\theta(n-1)$. A cursory glance at Fig. 8 shows that no duplications arise by those changes. In $\theta_{\mathscr{C}}(n+1) = \{1, s+2, 2, s+3, \ldots\}$ $\{s+2\}$ is replaced by $\{n, s+1\}$ and the last element is no longer included. Neither of $\{n, s+1\}$ was present before because $\lfloor \frac{1}{2}(s+1) \rfloor < s+1$ and $s + \lceil \frac{1}{2}(s+1) \rceil \le n$. As for the remaining overlap sets (farther to the right), most consist of $s+1$ consecutive elements of $t_{n,s}$. Due to the dropping of $\{s, n\}$ (see Fig. 8), one of the sets may consist of $s+1$ out of $s+3$ consecutive terms to $t_{n,s}$. By Lemma 4, any $2s+1$ consecutive terms of $t_{n,s}$ are distinct. Since $s \geq 2$ here, we conclude that *all* the overlap sets have size $s+1$, which completes the proof.

We illustrate the construction in subcase *a* with a (13, 8)-frame. Here $t=5$ and $s=3$. The fact that the last undistinguished position is the next-to-last position in σ_0 is an accident to the fact that $t=s+2$.

1 2 3 4 5 6 7 8

1 3 5 2 4 5 3 5 6 4 6 7 5 7 8 6 8 1 7 3 2 8 2 3 1

⊕ 9 10 11 12 13

1 8 4 2 6 3 7 4 8 5 1 6 2 7 4 1

Case b: $\Lambda_{m,s} \not\subseteq D_0$, *i.e.*, $s \mid 2(n-1)$, *but* $s>2$. (Also $s \neq \frac{1}{2}(n-1)$, so (9, 7) is excluded. Note that this case does not apply for any tight pair with $n \leq 8$.)

Starting with the construction of the frame in Case a, we need only modify the two braces that would share an endpoint at the now-undistinguished position $ns+l-2$. We replace those two braces with four braces by replacing the unavailable endpoint $ns+l-2$ by the endpoints $ns+l-3$, $ns+l-1$, and $ns+l$. Technically,

$$E_0 = \Lambda_{n,s} - \{ns+l-2\} \cup \{ns+l-3, ns+l-1, ns+l\}.$$

If in the previous construction the two braces ending at position $ns+l-2$ were the $(\beta-1)$th and the βth, receiving the labels $\beta-1$ and β, we assign to the four new braces the lables $\beta-1, \beta, \beta-1, \beta$, in order. Technically, we define $\mathscr{C}_0$ be setting $S_\alpha = \{\alpha\}$ for $1 \leq \alpha < \beta-1$, $S_{\beta-1} = \{\beta-1, \beta+1\}$, $S_\beta = \{\beta, \beta+2\}$ and $S_\alpha = \{\alpha+2\}$ for $\beta+1 \leq \alpha \leq m$.

By the same argument as before, this $(\sigma_0, D_0, E_0, \mathscr{C}_0)$ is (m,n)-complete and $\sigma_{\mathscr{C}_0}(\alpha) = t$ for $\alpha \neq \beta-1, \beta$. To show $\delta_{\mathscr{C}_0}(\beta) = \delta_{\mathscr{C}_0}(\beta-1) = t$ we must show that $|\theta_{\mathscr{C}_0}(\beta)| = |\theta_{\mathscr{C}_0}(\beta-1)| = s+2$, since this time *two* braces instead of one are used. The lengths of the braces were chosen so that $\theta(\beta)$ and $\theta(\beta-1)$ each consists of $s+2$ terms from σ_0. (See Fig. 9 for an example.) To show that these terms are distinct, we consider their location in $t_{n,s}$. The terms s and n were dropped and $\theta(\beta-1)$ does not include the now-distinguished term $s+1$. (See Fig. 9.) This means that $\theta(\beta)$ and $\theta(\beta-1)$ consist of $s+2$ out of $s+4$ or $s+5$ consecutive terms to $t_{n,s}$. Since $s \geq 3$, we have $2s+1 \geq s+4$. By Lemma 4, the only possible duplication is between the first and last terms in $\theta(\beta-1)$ when $s=3$, $n \equiv 1 \bmod 3$. In that case, this portion of the frame looks like

$\beta-1$ β $\beta-1$ β

$n-2$ 2 $n-1$ 4 1 $s+3$ 2

As remarked earlier Case b requires $n \geq 9$. Hence $n-2 \neq s+3$ and the proof of this case is finished.

We illustrate Case b with a (19, 11)-frame, where $t=7$ and $s=4$. See Fig. 9. (This case also applies to the pair (23, 10).)

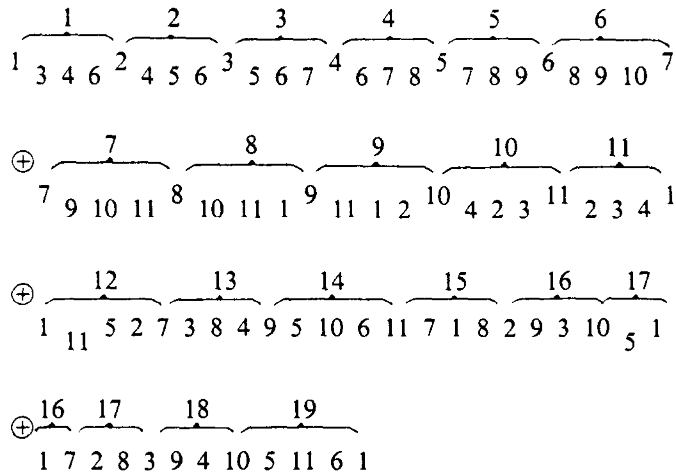
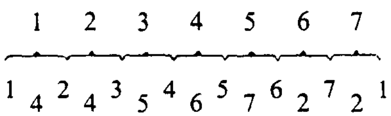


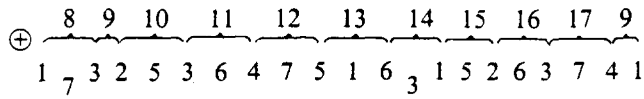
Fig. 9.

Case c: $s=2$, *so* $ns+l-2 \equiv 1 \pmod{s}$.

Again we start with the frame constructed in Case a but get stuck because $ns+l-2$ is the endpoint of braces. Here each brace has length 2, so the overlap sets have size 3. We want to shift the latter braces by 1 so as to straddle position $ns+l-2$ (see Fig. 10). This will require assigning one label to two shorter braces with an overlap set of size 4. Technically, we let E_0 consist of all the odd positions from 1 to $ns+3$, all the even positions from $ns+4$ to $nt-1$ and the final position nt. So the two short braces are $(ns+3, ns+4)$ and $(nt-1, nt)$, the $(n+2)$nd and $(m+1)$st, respectively. We define $\mathscr{C}_0$ be setting $S_\alpha = \{\alpha\}$ for $\alpha \neq n+2$ and $S_{n+2} = \{n+2, m+1\}$. As in Case a, $(\sigma_0, D_0, E_0, \mathscr{C}_0)$ is (m,n)-complete and the arguments used there show that $\delta_{\mathscr{C}_0}(\alpha) = t$ for $\alpha \neq n+2$. To complete the proof of this case, we need only show that $\theta_{\mathscr{C}_0}(n+2)$ consists of four distinct elements. A quick glance at Fig. 8 shows that they are, in order, 3, 2, $\lfloor \frac{1}{2}(n+1) \rfloor$ and 1. This follows since the last part of σ_0 is the last part of $t_{n,s}$, which is $\sigma_n, \lfloor \frac{1}{2}(n+1) \rfloor$ and the last two elements of $\sigma_{n,r}$ are $r+1$ and 1. As long as $n \geq 7$, these elements are distinct. The fact that they yield a duplication when $n=5$ is another indication that (7, 5) is a pathological case.

We demonstrate this construction with a (17, 7)-frame.





Since Cases a, b, and c together handle any tight pair except (n^2-n-1,n) and $(7,5)$, the proof of the theorem is complete. □

We find it interesting that although we have found several alternate constructions for (m,n)-complete frames for these cases for particular values of m and n, we have found no construction that works for all tight pairs or even for Cases b and c together.

6. The exceptional pairs (n^2-n-1,n) and $(7,5)$

We now turn our attention to the negative results. In order to simplify the arguments, we will find it convenient to introduce some additional notation and terminology. Recall that if F is a t-interval representation of a graph G, then for each $v \in V(G)$, $F(x)$ is the union of t_x pairwise disjoint closed intervals, where $1 \le t_x \le t$. Note that F can also be considered to be an interval representation of a graph on no more than $t \cdot |V(G)|$ vertices. We will call this interval graph G_F.

Recall that a graph without loops or multiple edges is a *forest* when it contains no cycles and that a *tree* is a connected forest.

Lemma 7. *If F is a t-interval representation of a bipartite graph G, then G_F is a forest.*

Proof. G_F is a triangle-free since G is bipartite. G_F is an interval graph and therefore contains no chordless cycles on four or more vertices and thus G_F is cycle-free.

Lemma 8. *Let (m,n) be tight and let $t=i(m,n)$. If F is a t-interval representation of $G=K(m,n)$, then the following statements hold:*

(i) *G_F is a tree.*

(ii) *$t_x=t$ for every vertex $x \in G$.*

(iii) *For each edge $\{x,y\} \in E(G)$ in F there is exactly one interval in $F(x)$ which intersects an interval in $F(y)$.*

Proof. Let e be the number of edges of G_F. Then since G_F is a forest, $e+1 \le (m+nm)t$ and, since all edges of G are represented, $e \ge mn$. But (m,n) tight implies $(m+n)t=mn+1$ and therefore equality holds in both of the previous statements. □

If F is a t-interval representation of $K(m, n \cdot p)$, let F_i be those intervals of F corresponding to vertices of $A \cup B_i$ for $1 \le i \le p$. Let F_i^* be obtained from F_i by collapsing any interval which intersects only one interval to a single point (except those at the left and right ends). Now identify the (m, n)-frame for F_i^* with F_i. Clearly if p is sufficiently large there exist integers i and j so that F_i and F_j determine the same frame.

We can now present a short proof of Theorem 5.

Theorem 5. *If $n \ge 3$ and $m = n^2 - n - 1$, then the interval number of the complete multipartite graph $K(m, n \cdot \infty)$ exceeds the interval number of the corresponding bipartite graph $K(m, n)$ by 1.*

Proof. Let $n \ge 3$ and $m = n^2 - n - 1$. We assume that $i(m, n \cdot \infty) = i(m, n)$ and proceed to obtain a contradiction. Note that $t = i(m, n) = n - 1$ and $s = n - t = 1$.

Let F be a t-interval representation of $K(m, n \cdot \infty)$. Assume that $K(m, n \cdot \infty)$ has been labeled so that F_1 and F_2 determine the same frame.

Since $n - 1$ degenerate intervals at Level 0 corresponding to a_α cannot intersect n intervals from level 1 or 2, there must be a nondegenerate interval for a_α for each $\alpha = 1, 2, \ldots, m$. The corresponding $n^2 - n - 1$ 'braces' overlap at least $n^2 - n$ intervals from Level 1. Since there are only $n^2 - n$ intervals at Level 1, all of them must be nondegenerate.

Now let us restrict our attention to the intersections between intervals corresponding to vertices of B_1 and B_2. Within a level 0 nondegenerate interval, at most one intersection of the form (j_1, j_2) can occur for $j_1 \ne j_2$. By Lemma 3, no such intersections can occur between nondegenerate intervals from Level 0, else F_1 or F_2 would be disconnected. Thus at most $n^2 - n - 1$ intersections of that form can occur. Since there are $n(n-1)$ such edges that must be represented, we conclude that F cannot be a t-interval representation of $K(m, n \cdot \infty)$. □

Note that Lemma 8 enabled us to avoid proving that $K(n, n \cdot \infty)$ has a t-interval representation only if there is an (m, n)-complete frame having density t. We continue that approach for the pathological case $(m, n) = (7, 5)$, which we tackle next.

Theorem 8. *The interval number of the complete multipartite graph $K(7, 5 \cdot \infty)$ exceeds the interval number of the corresponding complete bipartite graph by 1.*

Proof. For clarity we will occasionally refer to 7 and 5 as m and n. As in the proof of Theorem 7, we will assume $i(7, 5 \cdot \infty) = i(7, 5) = 3$ and proceed to obtain a contradiction. To that end let F be a 3-interval representation of $K(7, 5 \cdot \infty)$ and assume that $K(7, 5 \cdot \infty)$ has been labeled so that F_1 and F_2 determine the same frame. For ease of discussion, let $(\sigma, D, E, \mathscr{C})$ be the corresponding 4-tuple.

Again, we notice that 3 degenerate intervals at Level 0 cannot intersect 5 intervals from Level 1 and thus for each $\alpha = 1, 2, \ldots, m$ there is at least one distinguished brace in the frame, i.e. $|S_\alpha| \leq 1$. Note that when there are exactly k braces in S_α, the overlap set $\theta(\alpha)$ must contain at least $k+3$ different members to obtain $\delta_6(\alpha)=3$. In light of the 'tightness' described in Lemma 8, $\theta(\alpha)$ consists of only $k+2$ distinct members. Therefore when S_α contains one element, the corresponding brace overlaps three terms of σ, when S_α contains two elements the corresponding braces overlap two terms each, and never are the three intervals corresponding to a_α all nondegenerate. This quickly reduces to two cases, each of which will yield a contradiction.

We first consider the case where for each $\alpha = 1, 2, \ldots, m$, a_α appears in exactly one distinguished position at Level 0, i.e. $|S_\alpha| = 1$. Then there are eight distinguished positions in (σ, D) selected by E, and one position between each pair of them. This accounts for all 15 positions in (σ, D).

Now we consider the intersections between intervals representing B_1 and B_2. As before, we concentrate on intersections of the form (b_{1j_1}, b_{2j_2}) where $j_1 \neq j_2$, recognizing that no such intersections can occur between nondegenerate intervals from Level 0. Within a nondegenerate interval at Level 0, the corresponding portion of the frame is

$$\overbrace{i \quad j \quad k}^{\alpha}$$

The largest set of edges between B_1 and B_2 that can be represented by intersections of the corresponding intervals on Level 1 and 2 is $\{(b_{1i}, b_{2j}), (b_{1i}, b_{2k}), (b_{1j}, b_{2k})\}$ or dually $\{(b_{1k}, b_{2j}), (b_{1k}, b_{2i}), (b_{1j}, b_{2i})\}$. It is possible that not all three of these intersections occur. Some subset of the ones listed or the sets $\{(b_{1i}, b_{2j}), (b_{1k}, b_{2j})\}$ or $\{(b_{1j}, b_{2i}), (b_{1j}, b_{2k})\}$ are also possible. In any case with seven such braces at most 21 distinct edges of this type can occur. Since we need only $5 \cdot 4 = 20$, it would seem we might be able to complete the construction. To defeat it, we need to look more closely at the integers occupying distinguished positions in (σ, D), other than 1 and $l(\sigma)$. We will refer to such positions as *interior* distinguished positions. Consider those integers appearing only in interior distinguished positions of (σ, D). There are at most four such integers and at least 6 distinguished positions other than 1 and $l(\sigma)$. Thus, by the pigeonhole principle, some integer j_0 not occurring as $\sigma(1)$ or $\sigma(l(\sigma))$ occupies two different distinguished positions of σ. These positions must of course be nonconsecutive and therefore 4 braces use j_0 as an endpoint. Within each of these braces at most 1 intersection does not involve b_{1j_0} or b_{2j_0}. Within the remaining 3 braces, omitting the intersections that would use the interval corresponding to the j_0 in an undistinguished position, a total of $9-2=7$ such intersections are possible. We conclude that there are at most 11 intersections of the form (b_{1i}, b_{2j}) where $i \neq j$ and neither i nor j is j_0, but $4 \cdot 3 = 12$ such pairs (i, j) are required.

This leaves the case that at least one a_α has 2 nondegenerate positions. If there are k such a_α, then there are $7+k$ nondegenerate intervals at Level 0, with $7-k$

covering 3 terms each at Level 1 (or 2), and $2k$ covering two terms each. There must be $7+k+1$ distinguished positions in (σ, D) selected by E. The $7-k$ other terms appear under the brackets corresponding to a_α with $|S_\alpha|=1$. As before, we get at most 3 edges of the form (b_{1i}, b_{2j}) for $j \neq i$ represented for each of the $7-k$ long braces and we get at most one such edge with each of the remaining $2k$ braces for a total of $3(7-k)+2k=21-k$. Again 20 are needed, hence $k=1$ and within each of the $7-k$ braces three intersections occur.

Let α_0 be the unique integer such that a_{α_0} has two nondegenerate intervals. As noted above, the braces corresponding to a_{α_0} overlap four distinct integers in distinguished positions in (σ, D). Let $\{p, q\}$ be two of them that appear in interior distinguished positions.

The proof is completed by the following dilemma. If either of $\{p, q\}$ appears in another interior distinguished position, then not enough edges (b_{1i}, b_{2j}), $i \neq j$, not involving that index can be represented. But, if either of $\{p, q\}$ does not occur in another interior distinguished position, then not enough edges involving that index can arise. In the former case, at most $20-4-3-2=11$ intersections remain to represent the $4 \cdot 3=12$ other edges (4 for the other interior distinguished position, 3 for this one, and 2 for the remaining position). In the latter case, at most $3+2+2=7$ intersections can involve the label in question (3 for this distinguished position, 2 each for the other appearances), but we need $4+4=8$ for each. This completes the proof. □

7. Concluding remarks

Several natural problems involving interval numbers of complete multipartite numbers are motivated by the results of this paper.

(1) When $i(m, n \cdot \infty) = i(m, n)$ and (m, n) is tight, classify trees T which arise from optimal completely regular representations of $K(m, n \cdot \infty)$.

(2) More generally, classify all optimal completely regular representations of $K(m, n \cdot \infty)$.

(3) When $i(m, n \cdot \infty) = 1 + i(m, n)$, find the least value of p so that $i(m, n \cdot p) = 1 + i(m, n)$.

(4) Find other classes of graphs that admit to similar analysis in the sense that they have optimal representation sufficiently regular to permit rapid computations and explicit constructions.

(5) Develop and analyze algorithms for producing optimal or nearly optimal representations for graphs which model large systems.

(6) Extend these concepts to permit 'orderings' on tasks or weights on schedules.

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