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Arithmetic Progressions in Partially Ordered Sets

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Abstract. Van der Waerden's arithmetic sequence theorem – in particular, the 'density version' of Szemerédi – is generalized to partially ordered sets in the following manner. Let w and t be fixed positive integers and $\varepsilon > 0$. Then for every sufficiently large partially ordered set P of width at most w, every subset S of P satisfying $|S| \geqslant \varepsilon |P|$ contains a chain $a_1, a_2, \ldots, a_t$ such that the cardinality of the interval $[a_i, a_{i+1}]$ in P is the same for each i.

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1. Introduction

The primary objective in what follows is to extend the concept of arithmetic progression to (partially) ordered sets and to investigate conditions which force subsets to contain long arithmetic progressions. We call a sequence of distinct elements $a_1, a_2, \ldots, a_t$ from an ordered set P a *t-term arithmetic progression* in P if there is a positive integer d so that for each $i = 1, 2, \ldots, t-1$ the number of points in the closed interval $[a_i, a_{i+1}] = \{x \in P : a_i \leqslant x \leqslant a_{i+1}\}$ is precisely d. In particular, $a_1 < a_2 < \cdots < a_t$ must be a chain of P.

With this generalized notion of arithmetic progression we can extend the famous theorem of E. Szemerédi [3] which states that any subset of the positive integers having positive upper density contains arbitrarily long arithmetic progressions. Szemerédi's theorem is itself a strengthening of the following 1927 theorem of Van der Wearden:

THEOREM 1 (Van der Waerden [5]). *In any coloring of the positive integers*

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by finitely many colors, some color class must contain arbitrarily long arithmetic progressions. □

For our purposes the following finite version of Szemerédi's theorem is the most appropriate:

THEOREM 2 (Szemerédi [3]). *For every* $\varepsilon > 0$ *and every positive integer* t, *there exists* $N = N(\varepsilon, t)$ *so that if* S *is any subset of* $\{1, 2, \ldots, n\}$ *with* $n \geqslant N$ *and* $|S| \geqslant \varepsilon n$, *then* S *contains a* t-*term arithmetic progression.* □

Recall that an *antichain* in an ordered set is a set of pairwise incomparable elements, and that the *width* of a finite ordered set P is the cardinality of a largest antichain in P. We will prove the following generalization of Szemerédi's theorem. Note that the bound on width cannot be omitted, else we could not even guarantee the existence of a long chain.

THEOREM 3. *For every* $\varepsilon > 0$ *and every pair* t, w *of positive integers, there exists* $N = N(\varepsilon, t, w)$ *so that if* P *is any ordered set with* $|P| \geqslant N$ *and* $width(P) \leqslant w$, *and if* S *is any subset of* P *with* $|S| \geqslant \varepsilon |P|$, *then* S *contains a* t-*term arithmetic progression.* □

Before proceeding to the proof of Theorem 3, we pause to comment on another natural way to extend the concept of arithmetic progression to partially ordered sets. We could have required that the intervals $[a_i, a_{i+1}]$ be *isomorphic* as ordered sets, instead of equinumerous. It turns out, however, that this notion is too strong even to salvage a generalization of Theorem 1. The following example illustrates the dilemma.

In 1906, Thue [4] showed that there exist infinite square-free words on a three-letter alphabet, e.g., a sequence $X = (x_1, x_2, \cdots)$ drawn from $\{1, 2, 3\}$ with the property that for every $i, k \geqslant 1$, $(x_i, x_{i+1}, \ldots, x_{i+k-1}) \neq (x_{i+k}, x_{i+k+1}, \ldots, x_{i+2k-1})$. (For a more modern reference, see [1].)

Let Q_1, Q_2, and Q_3 be the black-and-white-colored ordered sets whose Hasse diagrams are pictured in Figure 1.

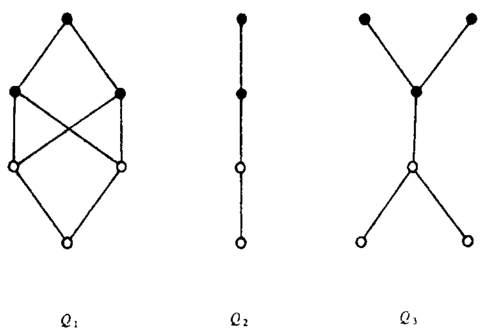

Fig. 1.

Let $P \oplus Q$ denote the *ordinal sum* of the ordered sets P and Q, i.e. the disjoint union of P and Q with the relations in P and in Q together with $p \leqslant q$ whenever $p \in P$ and $q \in Q$. Let P be the infinite, black-and-white-colored ordered set $P_1 \oplus P_2 \oplus \cdots$, where each P_i is a copy of Q_{x_i}. Then P has width 2, and we claim it has no monochromatic 3-term arithmetic progression.

To see this suppose $a_1 < a_2 < a_3$ is an arithmetic progression all three points of which are, say, white. Let r_i be the *rank* of a_i, that is, the length of the longest chain in P whose maximum elements is a_i; then r_1, r_2 and r_3 are either all congruent to 1 modulo 4 or all congruent to 2 modulo 4. However, the isomorphism class of the 'up-set' $\{y \in Q_j : y \geqslant x\}$ uniquely determines j whenever x is of rank 1 or 2 in Q_j. It follows that the intervals $[a_1, a_2]$ and $[a_2, a_3]$ determine matching, consecutive subsequences of X (each of length $(r_2 - r_1)/4$) contradicting the choice of X as a square-free sequence. (The argument for black arithmetic progressions is similar but uses down-sets.)

2. Proof of the Principal Theorem

We proceed by induction on w; the case $w = 1$ is Theorem 2, so we assume $w > 1$ and that Theorem 3 holds for all smaller values of w.

Let $\varepsilon > 0$ and let t be a fixed positive integer. Set $m = N(\varepsilon/5w, t, w-1)$, $b = \lceil 4wm/\varepsilon \rceil$, and $n_0 = \max(\lceil 4/\varepsilon^2 \rceil, N(\varepsilon/2b^2, t, 1))$. We will show that $N(\varepsilon, t, w)$ can be taken to be $n_0 b$.

Accordingly, let P be an arbitrary ordered set with $|P| \geqslant N = n_0 b$ and width$(P) \leqslant w$, and let S be an subset of P for which $|S| \geqslant \varepsilon |P|$. We claim that S contains a t-term arithmetic progression.

Using Dilworth's Theorem, we partition P into w chains $P = C_1 \cup C_2 \cup \cdots \cup C_w$. We then determine a second partition $P = B_1 \cup B_2 \cup \cdots \cup B_{n+1}$ into subsets which we call *blocks*. This partition must satisfy the following two conditions:

(1) $|B_j| = b$ for $j = 1, 2, \ldots, n$ and $|B_{n+1}| < b$ (thus $n \geqslant n_0$);

(2) If $1 \leqslant i < j \leqslant n+1$, $x \in B_i$ and $y \in B_j$, then $y \nless x$ in P.

Such a partition can be constructed easily enough from the bottom up by inductively assigning minimal elements to successive blocks.

We say that a block B_i, $1 \leqslant i \leqslant n$, is *dense* if the density of S in B_i is at least $\varepsilon/2$, i.e., $|S \cap B_i| \geqslant \varepsilon b/2 \geqslant 2wm$. Note that there are at least $\varepsilon n/2$ dense blocks, for otherwise we could conclude that $|S| < (\varepsilon n/2)b + (n - \varepsilon n/2)\varepsilon b/2 + b < \varepsilon bn \leqslant \varepsilon |P|$. For each dense block B_i, we now describe a process by which a point $y_i \in B_i \cap S$, called the *root* of B_i, will be selected. The root y_i and an associated pair (r_i, s_i) of integers are determined as follows.

Since $|B_i \cap S| \geqslant 2wm$ we may choose a chain C_j from the chain partition so that $|C_j \cap B_i \cap S| \geqslant 2m$. Let x_i be the least element, and z_i the greatest element, of $C_j \cap B_i \cap S$. Then choose y_i so that $[x_i, y_i] \cap S$ and $[y_i, z_i] \cap S$ each contain at least m points of S. Finally, let r_i and s_i count the points below and above y_i in the block, i.e., $r_i = |\{u \in B_i : u \leqslant y_i\}|$ and $s_i = |\{v \in B_i : v \geqslant y_i\}|$.

If in some dense block B_i the interval $[y_i, z_i]$ has width at most $w-1$, then since the density of S in $[y_i, z_i]$ is at least $m/b > \varepsilon/5w$ while $m = N(\varepsilon/5w, t, w-1)$, $[y_i, z_i]$ already contains the desired t-term arithmetic progression. We may thus assume that $[y_i, z_i]$ has full width w, and hence intersects every chain C_k of the chain partition in at least one element, say u_k, of B_i. If v is an element of some block B_q with $i < q \leqslant n$ then $v \in C_k$ for some k, hence $v > u_k \geqslant y_i$.

After dualizing the above argument using the intervals $[x_i, y_i]$, we may assume that the root of every dense block is comparable to every element of P outside that block. The situation in a typical dense block is illustrated in Figure 2, where the crooked vertical lines represent chains of the chain partition and the circular arcs enclose intervals; the set S and the covering relations of P are not pictured.

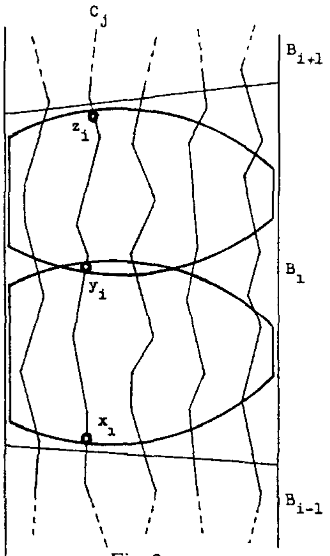

Fig. 2.

For each r, s such that $1 \leqslant r \leqslant b$ and $1 \leqslant s \leqslant b$, let $U_{r,s} = \{i : B_i \text{ is dense}, r_i = r \text{ and } s_i = s\}$. Since as noted above there are at least $\varepsilon n/2$ dense blocks, there must be values of r and s for which $|U_{r,s}| \geqslant \varepsilon n/2b^2$. Recalling that $n = N(\varepsilon/2b^2, t, 1)$ and applying Theorem 2 to the chain of values of i, we obtain a t-term arithmetic progression $i_1, i_2, \ldots, i_t$ within $U_{r,s}$. Let d be the common difference $i_{j+1} - i_j$.

It follows that the corresponding roots $y_{i_1}, y_{i_2}, \ldots, y_{i_t}$ form a t-term arithmetic progression in P, because each interval $[y_{i_j}, y_{i_{j+1}}]$ contains exactly $s + (d-1)b + r$ points of P. Since the roots all lie in S the proof of the theorem is complete. □

3. Concluding Remarks and Problems

Any ε-dense subset of a large, width-bounded ordered set will contain a long chain C. The function α: $C^2 \to \{1, 2, \ldots\}$ given by $\alpha(x, y) = |[x, y]|$ is (essentially) superadditive, that is, $x < y < z$ implies $\alpha(x, z) \geqslant \alpha(x, y) + \alpha(y, z) - 1$. One might thus be led to ask whether *any* such function leads to a Szemerédi-type theorem: if such a function β is defined on a long chain, say $\{1 < 2 < \cdots < n\}$, and $\beta(1, n)/n$ is bounded in advance, must there be a long subchain $i_1, i_2, \ldots, i_t$ for which $\beta(i_j, i_{j+1})$ is constant?

It turns out that this will not generally hold. For example, let $\beta(i, j) = 3(j - 2) - 1$ if i and j are elements number k and $k + 1$ of an ordinary arithmetic progression beginning with a member of $[1, j - i]$, and k is *odd*; otherwise $\beta(i, j) = 3(j - i) - 2$. It is easily seen that then $\beta(i_1, i_2) \neq \beta(i_2, i_3)$ whenever $i_1 < i_2 < i_3$.

Since the structure of partially ordered sets does seem to play a role, it is perhaps worth investigating other order-theoretic functions. Two possibilities are $h(x, y) =$ the *height* of $[x, y]$, i.e., the length of the longest chain in P from x to y, and $\lambda(x, y) =$ the length of the shortest *covering* chain from x to y. Each leads to another generalization of the notion of arithmetic progression. In the case of $h(x, y)$, Van der Waerden's theorem generalizes easily since P contains a long maximum-length chain, on which the theorem for positive integers can directly be applied. However, it is not obvious to us that the density version holds, and neither version is apparent in the case of the subadditive function $\lambda(x, y)$.

Returning to our original definition of arithmetic progression, we ask now whether the bound on width can be replaced by other algebraic or combinatorial properties for partially ordered sets which force subsets of positive density to contain long arithmetic progressions. For example, J. Walker [6] pointed out that, as an immediate consequence of Szemerédi's theorem, a positive fraction of the set of subsets of a large set, ordered by inclusion, contains long arithmetic progressions. We suspect that this phenomenon may in fact hold for general distributive lattices.

CONJECTURE. For every $\varepsilon > 0$ and every positive integer t, there exists $M = M(\varepsilon, t)$ so that if L is a distributive lattice of cardinality at least M, and S is a subset of L with $|S| \geqslant \varepsilon |L|$, then S contains a t-term arithmetic progression. □

Although we have made no real progress in establishing this conjecture, we comment that the following result of L. Kahn and M. Saks (restated in our terms) is an initial step:

THEOREM 4 (Kahn and Saks [2]). *For every $\varepsilon > 0$, there exists $M = M(\varepsilon)$ so that if L is a distributive lattice of cardinality at least M and S is any subset of L with $|S| \geqslant \varepsilon |L|$, then S is not an antichain.* □

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