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NOTE

POSET BOXICITY OF GRAPHS

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A *t-box representation* of a graph encodes each vertex as a box in t-space determined by the (integer) coordinates of its lower and upper corner, such that vertices are adjacent if and only if the corresponding boxes intersect. The *boxicity* of a graph G is the minimum t for which this can be done; equivalently, it is the minimum t such that G can be expressed as the intersection graph of intervals in the t-dimensional poset that is the product of t chains. Scheinerman defined the *poset boxicity* of a graph G to be the minimum t such that G is the intersection graph of intervals in some t-dimensional poset. In this paper, a special class of posets is used to show that the poset boxicity of a graph on n points is at most $O(\log\log n)$. Furthermore, Ramsey's theorem is used to show the existence of graphs with arbitrarily large poset boxicity.

1. Introduction

"Boxicity" is a representation parameter of graphs introduced by Roberts [2] and Cohen [1]. It is the minimum dimension in which the graph can be represented as an intersection graph of boxes with sides parallel to the axes. More precisely, a *t-box representation* of a graph encodes each vertex as a box in t-space determined by the (integer) coordinates of its lower and upper corner, such that vertices are adjacent if and only if the corresponding boxes intersect. The *boxicity* of a graph G is the minimum t for which this can be done. Since it can be assumed that the upper and lower coordinates are all integers, a t-box representation expresses G as an intersection graph of intervals in the t-dimensional poset that is the product of t chains. Scheinerman [3] defined the *poset boxicity* of a graph G to be the minimum t such that G is the intersection graph of intervals in a t-dimensional poset. (A general discussion of representation parameters of graphs, included the results mentioned here, appears in [6].)

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In this paper, we consider how large the poset boxicity can be for a graph on n points. The best possible upper bound for boxicity is $\lfloor \frac{1}{2}n \rfloor$ [2], with the extremal graphs characterized in [5]. The only graph achieving boxicity $\frac{1}{2}n$ is $K_{2,\ldots,2}$, but the poset boxicity of this graph is always at most 4. We will construct a family of graphs whose poset boxicity cannot be bounded by any constant, which we show by repeated application of Ramsey's Theorem. First, we use a special class of posets to show that the poset boxicity of a graph on n points is always at most $O(\log \log n)$.

2. The upper bound

Theorem 1. *The poset boxicity of a graph on n vertices is at most* $O(\log \log n)$.

Proof. Given G on n vertices, we define a poset $p(G)$ of height 2. $P(G)$ has a maximal element a_i and a minimal element b_i for each vertex v_i in G. $P(G)$ has a middle element c_e for each edge e in G, and the relations are defined by $a_i > c_e$ and $b_i < c_e$ if and only if $i \in e$. For simplicity, we also have $a_i > b_j$ for all i, j. Clearly G is the intersection graph of the intervals $\{(a_i, b_i)\}$ in $P(G)$; the intervals intersect if and only if G has the edge v_iv_j.

The dimension of $p(G)$ is at most twice the dimension of the poset Q induced by its middle and bottom levels, because any realizer L for Q can be extended to a realizer for P by taking two copies L_1 and L_2, upside-down, replacing each appearance of b_i in L_2 by a_i, adding $a_1, \ldots a_n$ at the top of each chain of L_1, and adding $b_1, \ldots, b_n$ at the bottom of each chain of the modified L_2. Hence we consider Q. For any G, the resulting Q is a subposet of the poset induced by the sets of size 1 and 2 among the lattice of all subsets of an n-set. Hence its dimension is at most the dimension of that poset. Spencer [4] showed that the dimension of that poset is $O(\log \log n)$. □

3. The lower bound

Theorem 2. *For any integer t, there exists a graph whose poset boxicity exceeds t.*

Proof. Suppose that every graph can be represented in a t-dimensional poset. Consider a graph G_n defined on the 2-element subsets of $\{1, \ldots, n\}$ by creating an edge between $\{i, j\}$ and $\{j, k\}$ for each triple $i < j < k$. Let P be a poset of dimension at most t in which G has an interval representation, and let $I(i, j)$ be the interval of P assigned to the vertex $\{i, j\}$ by the representation. Let $a(i, j)$ and $b(i, j)$ be the top and bottom elements of $I(i, j)$. For each triple $i < j < k$, choose an element $p(i, j, k) \in I(i, j) \cap I(j, k)$.

Now we define a 2-coloring on the 5-subsets of $\{1, \ldots, n\}$. Given a 5-set $i_1 < i_2 < i_3 < i_4 < i_5$, note that $p(i_1, i_3, i_5)$ cannot belong to $I(i_2, i_4)$, since there is no edge from $\{i_2, i_4\}$ to $\{i_1, i_3\}$ or $\{i_3, i_5\}$ in G_n. Hence $p(i_1, i_3, i_5)$ is not greater than $b(i_2, i_4)$ or is not less than $a(i_2, i_4)$. Color the 5-set "bottom" if $p(i_1, i_3, i_5)$ is not greater than $b(i_2, i_4)$; otherwise, color it "top". If n is sufficiently large, we can guarantee as large a set H as we desire all of whose 5-sets get the same color. By symmetry, we may suppose this color is "bottom".

Now we t-color the 5-sets of H. For each $\{i_1 < i_2 < i_3 < i_4 < i_5\}$ we know $p(i_1, i_3, i_5)$ is not greater than $b(i_2, i_4)$, so there is some extension L_j in the t-realizer for P such that $b(i_2, i_4)$ lies above $p(i_1, i_3, i_5)$ in L_j; give the 5-set a color corresponding to such an extension. If H is sufficiently large, then it has some 6-set $\{i_1 < i_2 < i_3 < i_4 < i_5 < i_6\}$ whose 5-sets all get the same color j. Applying the defining condition for color j to the 5-sets $\{i_1 < i_2 < i_3 < i_4 < i_5\}$ and $\{i_2 < i_3 < i_4 < i_5 < i_6\}$ yields $b(i_2, i_4) > p(i_1, i_3, i_5) \geqslant b(i_3, i_5) > p(i_2, i_4, i_6) \geqslant b(i_2, i_4)$ in L_j. This contradiction means that G_n cannot have an interval representation in a t-dimensional poset if n is sufficiently large. □

Let $R_s(k, \ldots, k)$ denote the Ramsey number for t-coloring s-sets to force a set of size k whose s-sets all get the same color. We have shown that if $n > R_5(M, M)$, where $M = R_5(6, \ldots, 6)$ (t colors), then the poset boxicity of G_n, a graph on $\binom{n}{2}$ vertices, exceeds t. This lower bound for worst-case poset boxicity of a graph on N vertices grows unimaginably slowly.

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