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Irreducible Posets with Large Height Exist

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The dimension of a poset (X, P) is the minimum number of linear extensions of P whose intersection is P. A poset is irreducible if the removal of any point lowers the dimension. If A is an antichain in X and $X - A \neq \varnothing$, then dim $X \leqslant 2$ width $(X - A) + 1$. We construct examples to show that this inequality is best possible; these examples prove the existence of irreducible posets of arbitrarily large height. Although many infinite families of irreducible posets are known, no explicity constructed irreducible poset of height larger than four has been found.

1. INTRODUCTION

A poset consists of a set X and a partial order P on X; the notations $(x, y) \in P$ and $x \leqslant y$ are used interchangeably. The dimension of (X, P), dim X, is the minimum number of linear extensions of P whose intersection is P [3]. Equivalently dim X is the smallest integer k such that X is isomorphic with a subposet of R^k [6]. We consider a linear extension L of X as a listing of X such that $x > y$ in P implies x over y in L. Then dim X is the smallest integer k such that there exists k linear extensions $L_1, L_2, ..., L_k$ such that for every incomparable pair $x, y \in X$, there exist integers i, j with x over y in L_i and y over x in L_j. A poset is irreducible if $\dim(X - x) < \dim X$ for every $x \in X$.

If A is an antichain of X and $X - A \neq \varnothing$, then dim $X \leqslant 2$ width $(X - A) + 1$. In this paper, we construct examples to show that this inequality is best possible. These examples show that for every integer h, there exists an irreducible poset whose height is greater than h.

2. AN ELEMENTARY INEQUALITY

THEOREM 1. *Let (X, P) be a poset and A an antichain. If $X - A \neq \varnothing$, then dim $X \leqslant 2$ width $(X - A) + 1$.*

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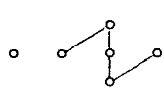
Proof. Let w = width of $X - A$. Then by Dilworth's theorem [2], there exists a decomposition $X - A = C_1 \cup \cdots \cup C_w$ where C_i is a chain. Furthermore it is well known [4] that for any chain C of a poset (X, P), there exists a linear extension L (called an upper extension) such that for every incomparable pair $x, y \in X$ with $x \in C$ and $y \notin C$, we have y over x in L. A similar statement can be made about the existence of lower extensions. Therefore for each $i \leqslant w$, let L_{2i-1} and L_{2i} be upper and lower extensions for the chain C_i.

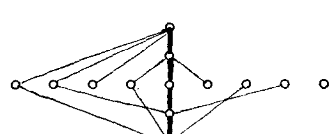
Let M_1 be the restriction of L_1 to A. Then there exists a linear extension L_{2w+1} of P such that the restriction of L_{2w+1} to A is $\hat{M}_1$ [1]. Finally we observe that $L_1 \cap L_2 \cap \cdots \cap L_{2w+1} = P$ since the first $2w$ extensions establish all incomparabilities except possibly those involving a pair of elements from A and this situation is handled by L_1 and L_{2w+1}.

3. Definition of $X(n, h)$

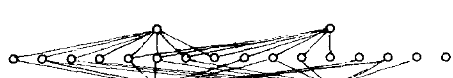
For each $n \geqslant 1$, $h \geqslant 1$ we define the poset denoted $X(n, h)$ as follows. A is a maximal antichain in $X(n, h)$ and $X(n, h) - A = X_U \cup X_L$ is the partitioning of the remaining points into upper and lower halves. X_U consists of n incomparable chains $C_1, C_2, \ldots, C_n$ and each chain C_i contains h points $c_{i1} > c_{i2} > \cdots > c_{ih}$. Similarly X_L consists of n incomparable chains $D_1, D_2, \ldots, D_n$ and each chain D_j contains h points $d_{j1} > d_{j2} > \cdots > d_{jh}$. Every point of X_U is over every point of X_L so that the width of $X(n, h) - A$ is n. The anitchain A then contains one point for each ordered pair (S, T) where S is an order ideal of $\hat{X}_U$ and T is an order ideal of X_L. The element of A corresponding to the pair (S, T) is less than all points in S and greater than all points in T. Therefore there are $(h + 1)^{2n}$ elements in A.



X(1,1)



X(1,2)



X(2,1)

Figure 1

We illustrate this definition with Hasse Diagrams for $X(1, 1)$, $X(1, 2)$, and $X(2, 1)$.

4. Some Inequalities for dim $X(n, h)$

In this section we discuss the behavior of dim $X(n, h)$ when one of the parameters is fixed and the other becomes large. In considering the proof given in Section 3, we see that $2n$ extensions are sufficient to establish all incomparabilities involving pairs from $X - A$ and those involving an incomparable pair x, a with $x \in X - A$ and $a \in A$. We first prove that if n is sufficiently large compared to h, we can also establish the incomparabilities involving pairs from A at the same time.

Theorem 2. *For each h, there exists a constant n_h such that for all $n \geqslant n_h$, $\dim X(n, h) \leqslant 2n$.*

Proof. Let $n_h = h + 1$ and suppose $n \geqslant n_h$. We will construct $2n$ lists of $X(n, h)$ which establish all incomparabilities. For each $i \leqslant h + 1$ we construct L_{2i-1} and L_{2i} as follows. To order the elements of $X - A$, first place all elements of C_i under the remaining elements of X_U; also place all elements of D_i over the remaining elements of X_L. Order the remaining elements of $X_U - C_i$ and $X_L - D_i$ in any order that is consistent with P.

We now describe a process for interpolating the elements of A into these lists. In each pair L_{2i-1} and L_{2i}, all elements of A will remain under all elements of $X_U - C_i$ and over all elements of $X_L - D_i$. With this restriction there are $2h + 1$ "gaps" in each list in which elements of A may be placed. In L_1 place all elements of A in the highest gap which the ordering P will permit, i.e., if $c \in C_1$, $a \in A$ and cIa in P, then c is under a in L_1. In L_2 place all elements of A in the lowest gap which the ordering P will permit. In L_2 order all elements of A which appear in the same gap arbitrarily. Then for each gap G in L_1, order the elements of G by the dual of the restriction of L_2 to G. This completes the description of L_1 and L_2. Note that elements of A appear in only $h + 1$ gaps in both L_1 and L_2. Let $A = G_1 \cup G_2 \cup \cdots \cup G_{h+1}$ be the partitioning of A into subsets consisting of elements which have been inserted in the same gap in L_1 with G_1 being the highest gap and G_{h+1} the lowest. Note that G_{h+1} consists of those elements of A which are under all elements of C_1. G_h consists of those elements of A which are incomparable with c_{1h} but less than the remaining elements of C_1, etc.

Then in L_3 put elements of G_2 over all elements of A–G_2. Then place all elements of G_2 over those elements of C_2 with which they are incomparable and put all elements of A–G_2 under those elements of D_2 with which they are incomparable. In L_4 put elements of G_2 under those elements of D_2 with which they are incomparable and elements of A–G_2 over those elements of C_2 with which they are incomparable. Elements of A not already ordered by this construction can then be ordered arbitrarily.

We continue in this fashion, first forcing G_i over A–G_i in L_{2i-1}. We then force G_i up in L_{2i-1} and down in L_{2i} while forcing A–G_i down in L_{2i-1} and up in L_{2i}.

For each $i \leqslant h + 1$ all elements of G_i are over all elements of A–G_i in L_{2i-1}. Thus incomparabilities between elements of A which belong to distinct G's are established. L_1 and L_2 already establish incomparabilities between elements of A which belong to the same G_i.

If $h + 1 < n$ for each i with $h + 1 < i \leqslant n$, let L_{2i-1} and L_{2i} be upper and lower extensions respectively of the chain $C_i \cup D_i$. Then it is clear that $L_1 \cap L_2 \cap \cdots \cap L_{2n} = P$ and dim $X(n, h) \leqslant 2n$.

The constant $n_h = h + 1$ used in this proof is not best possible. It is easy to see that $n_h = \{\log_2 (h + 1)\}$ will suffice. On the other hand it seems reasonable that n_h must increase with h and if h is very large compared to n, then an extra list to complete the task of establishing the incomparabilities among elements of A may be required. The proof that this is indeed true is somewhat more complicated than the preceding result. We first make the following definition. Let $\mathscr{C}$ be a collection of linear extensions of $X(n, h)$. We say that $\mathscr{C}$ satisfies property * if for every $c \in X_U$, $a \in A$ with $a\, I\, c$, there exists $L \in \mathscr{C}$ with a over c in L and for every $d \in X_L$, $a \in A$ with $d\, I\, a$ in P, there exists $L \in \mathscr{C}$ with d over a in L.

If $\mathscr{C}$ satisfies * for $X(n, h)$, then certain incomparabilities among elements of A are automatically established. The next result shows that there is a linear order on A which introduces no order on a pair of elements of A not already required by one of the lists in $\mathscr{C}$.

LEMMA 1. *Let $\mathscr{C}$ be a collection of linear extensions of $X(n, h)$ which satisfies property *. Then there exists a linear extension M of A such that a over a' in M implies that a is over a' in some $L \in \mathscr{C}$.*

Proof. We remember that there is an identification between elements of A and ordered pairs (S, T) where S is a order ideal of $\hat{X}_U$ and T is an order ideal of X_L.

There is a natural partial ordering Q on A defined by $(S, T) \leqslant (S', T')$ in Q if $S \subseteq S'$ and $T' \subseteq T$. Let M be any linear extension of Q and suppose $a = (S, T)$ is under $a' = (S', T')$ in M. Further suppose that a' is over a in every $L \in \mathscr{C}$. If $x \in S' - S$, then $a\, I\, x$ in P and thus a is over x in some L which implies that a is over a' in L. Similarly if $y \in T - T'$ then a is over a' in some $L \in \mathscr{C}$. The contradictions require that $S' \subseteq S$ and $T \subseteq T'$, i.e.,

$a' \leqslant a$ in Q. Since M is an extension of Q, a under a' in M is not possible and the proof of the lemma is complete.

LEMMA 2. *If L_1 and L_2 are a pair of linear extensions of $X(1, 2)$ which satisfy *, then there is a distinct pair $a, a' \in A$ with a under a' in both L_1 and L_2.*

Proof. Consider the following seven element subposet of $X(1, 2)$.

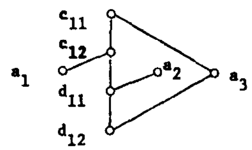


FIGURE 2

We may assume without loss of generality that a_3 is between c_{11} and c_{12} in L_1 and between d_{11} and d_{12} in L_2. Now a_2 must be over c_{11} in one of the lists; if a_2 is over c_{11} in L_1 then a_2 is over a_3 in both lists. Hence we may assume a_2 is over c_{11} in L_2. Similarly we may assume a_1 is under d_{12} in L_1 but this requires a_1 to be under a_2 in both lists.

The subposet of $X(1, 2)$ used in the proof of the preceding lemma is one of the 13 irreducible posets which have dimension three and seven points [8]. These posets are the only irreducible posets which have dimension n and $2n + 1$ points for any $n \geqslant 1$ [5].

LEMMA 3. *For every $n \geqslant 1$, there is a constant h_n' such that for all $h \geqslant h_n'$ and for every collection $\mathscr{C}$ of $2n$ linear extensions of $X(n, h)$ satisfying property *, there exists a distinct pair $a, a' \in A$ with a' in every $L \in \mathscr{C}$.*

Proof. The proof is by induction on n. Lemma 2 shows that the result holds when $n = 1$ and that a suitable choice for h_1' is 2. We assume that the result holds for all $n \leqslant k$ and suppose that $n = k + 1$. Let $m = 2 + (h_k' + 1)^{2k}$. We then show that a satisfactory choice for h'_{k+1} is $3m$. First note that $m > 2^{k+1} + 2$. Now since $h > h'_{k+1}$ implies $X(k + 1, h)$ contains $X(k + 1, h'_{k+1})$ as a subposet, we need only show that for every collection $\mathscr{C}$ of $2k + 2$ lists of $X(k + 1, 3m)$ which satisfies property *, there exists a distinct pair $a, a' \in A$ such that a is over a' in every $L \in \mathscr{C}$.

We consider each chain in X_U and X_L in $X(k + 1, 3m)$ as being partitioned into three sections, each consisting of m consecutive points; we refer to these as the top, middle, and bottom sections and denote them by C_i', C_i'', C_i''' in X_U and D_i', D_i'', D_i''' in X_L.

Suppose that for each $i \leqslant k + 1$, E_i is a subchain of C_i. Now fix some integer $i_0 \leqslant k+1$; then there is an element $a \in A$ which is less than the bottom element of each E_i with $i \neq i_0$ and incomparable with every point in E_{i_0}. Since $\mathscr{C}$ satisfies property *, there is some $L \in \mathscr{C}$ in which this a is over the highest element of E_{i_0}. In this linear extension, all points of the chain E_{i_0} are under the points of the other chains and we say E_{i_0} gets to the bottom of $\{E_i \mid i \leqslant k+1\}$ in L. Note that in some linear extensions belonging to $\mathscr{C}$, none of the chains in $\{E_i \mid i \leqslant k+1\}$ may be on the bottom; however each chain in $\{E_i \mid i \leqslant k+1\}$ gets to the bottom in at least one list in $\mathscr{C}$.

If F_i is a subchain of D_i for each $i \leqslant k+1$, similar reasoning shows that each chain in $\{F_i \mid i \leqslant k+1\}$ gets to the top in at least one list in $\mathscr{C}$.

At this point we consider the collections of middle chain $\{C_i'', \mid i \leqslant k+1\}$ and $\{D_i'' \mid i \leqslant k+1\}$. Suppose each chain in $\{C_i'' \mid i \leqslant k+1\}$ gets to the bottom twice and each chain in $\{D_i'' \mid i \leqslant k+1\}$ gets to the top twice.

Now we restrict our attention to those elements of A which are less than each of the top elements of the chains $\{C_i'' \mid i \leqslant k+1\}$ and greater than each of the bottom elements of the chains $\{D_i'' \mid i \leqslant k+1\}$. Of these elements, consider the subset $A' = \{a_j \mid 1 \leqslant j \leqslant 2^{k+1}+1\}$ where a_j is less than the top j elements of each C_i'' and incomparable with the remaining elements of C_i''. Each a_j is incomparable with the top j elements of each D_i'' and greater than the remaining elements. Each $a \in A'$ can be over elements from at most one chain of $\{C_i'' \mid i \leqslant k+1\}$ in any linear extension in $\mathscr{C}$. Therefore any $a \in A'$ goes up in $k+1$ lists and down in the remaining $k+1$ lists. Furthermore, when an element $a \in A'$ is over some elements of a middle chain C_i'' with which it is incomparable, it is over all elements of C_i'' with which it is incomparable. A similar statement holds for the middle chains of X_L.

Since $|A'| > 2^{k+1}$, there exists a pair of integers $j_1 < j_2$ such that a_{j_1} and a_{j_2} behave the same way in each linear extension of $\mathscr{C}$, i.e., a_{j_1} is over elements of C_i'' in L iff a_{j_2} is over elements of C_i'' in L (dually for the middle chains of X_L). But this implies that a_{j_1} is over a_{j_2} in every L in $\mathscr{C}$.

A second possibility is that all the chains of $\{C_i'' \mid i \leqslant k+1\}$ get to the bottom twice but that one of the middle chains of X_L, say D_{k+1}'', gets to the top of $\{D_i'' \mid i \leqslant k+1\}$ only once. We label the chains and the extensions in $\mathscr{C}$ so that D_{k+1}'' gets to the top of $\{D_i'' \mid i \leqslant k+1\}$ in L_{2k+2} and C_{k+1}'' gets to the bottom of $\{C_i'' \mid i \leqslant k+1\}$ in L_{2k+1} and L_{2k+2}.

Consider the subset A'' of A consisting of those points which are greater than the top elements of the chains of $D_1'', D_2'', \ldots, D_k''$ and less than the top elements of $C_1'', C_2'', \ldots, C_k''$. Notice that no element of A'' can be under an element of the bottom section of D_{k+1} in any linear extension except L_{2k+2}. Furthermore, any element of A'' which is incomparable with elements of D'''_{k+1} and C''_{k+1} must be over those elements of C''_{k+1} in L_{2k+1} and under those elements of D'''_{k+1} in L_{2k+2}.

Let M be a linear extension of $X(k, h_k')$ provided by Lemma 1. Now in the subposet of $X(k + 1, 3m)$ determined by

$$\{C_i'' \mid i \leqslant k\} \cup \{D_i' \mid i \leqslant k\} \cup A'',$$

there are many copies of $X(k, h_k')$. We choose one copy and call it Y by specifying the incomparabilities which the antichain of Y must have with elements of C''_{k+1} and D'''_{k+1}. If M orders the elements of the antichain in Y by $\{a_1, a_2, a_3, ...\}$, then we require each a_i to be less than the top i elements of C''_{k+1} and incomparable with the remaining elements of C''_{k+1}. We also require each a_i to be incomparable with the top i elements of D'''_{k+1} and greater than the remaining elements.

Then it is easy to see that the collection $\mathscr{D}$ consisting of the $2k$ linear extensions obtained by restricting $L_1, L_2, ..., L_{2k}$ to Y satisfies property * for this copy of $X(k, h_k')$. Thus there is a distinct pair a, a' from the antichain of Y which are ordered the same way by the restrictions of $L_1, L_2, ..., L_{2k}$ to Y. Since the restrictions of L_{2k+2} and L_{2k+2} to the antichain of Y are both M, it is clear that a and a' are in the same order in every $L \in \mathscr{C}$.

Hence we may assume that there is a middle chain which gets to the bottom of $\{C_i'' \mid i \leqslant n\}$ only once and a middle chain of X_L which gets to the top of $\{D_i'' \mid i \leqslant n\}$ only once. By an argument similar to the one just given, it is straightforward to show that the desired result follows and the proof of our lemma is complete.

If $\mathscr{C}$ is a collection of linear extensions of $X(n, h)$ whose intersection is the partial order on $X(n, h)$, then $\mathscr{C}$ satisfies property *. Thus we have proved the following.

THEOREM 3. *For every* $n \geqslant 1$, *there exists a constant* h_n *such that for all* $h \geqslant h_n$, $\dim X(n, h) = 2n + 1$.

Of course, this also shows that the bound given by Theorem 1 is best possible.

4. THE EXISTENCE OF IRREDUCIBLE POSETS WITH LARGE HEIGHT

For $n \geqslant 2$, it is an open problem to find the best value of the constant h_n of Theorem 3. An even more difficult problem is to find irreducible subposets of $X(n, h)$ with the same dimensions as $X(n, h)$. However it is clear that Theorems 2 and 3 together prove that irreducible posets of arbitrarily large height exist. At this time no explicitly constructed irreducible poset with height larger than 4 is known. An infinite family of irreducible posets of height 4 is given in [7].

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