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The Number of Depth-First Searches of an Ordered Set

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Abstract. We show that the problems of deciding whether an ordered set has at least k depth-first linear extensions and whether an ordered set has at least k greedy linear extensions are NP-hard.

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Key words. Depth-first linear extension, greedy linear extension, ordered set, NP-complete.

1. Introduction

In this article, we settle a problem posed by M. Habib [3] concerning the enumeration of depth-first and greedy linear extensions of an ordered set. Habib conjectured that it is NP-hard to say whether an ordered set has at most k greedy linear extensions, but that there should be a polynomial time algorithm for answering the same question for depth-first extensions. We show, however, that both problems are NP-hard.

Our proof techniques involve transforming well known NP-complete problems (satisfiability and vertex cover) into enumeration problems for ordered sets. These transformations are extensions of transformations which first appeared in Kierstead [5]. The rest of this section is devoted to the formal statements of our problems and the notation necessary for the transformations.

Our ordered set notation is quite standard; one exception is that we use $<:$ to denote the cover relation in an ordered set. Let L be a linear extension of an ordered set P defined by $x_1 < x_2 < \cdots < x_n$. For each $i = 1, 2, \ldots, n$, let $M_i = \mathrm{Min}(P - \{x_1, x_2, \ldots, x_i\})$, and let $T_i = \{x \in M_i : x_i <: x\}$. If $T_i \neq \emptyset$, set $G_i = T_i$; else set $G_i = M_i$. The extension L is *greedy* if $x_{i+1} \in G_i$ for all $i \geqslant 1$. We call the pair (x_i, x_{i+1}) a jump if $T_i = \emptyset$. Similarly, for each $i = 1, 2, \ldots, n$, let $S_i = \{j \leqslant i$: there exists $x \in M_i$ with $x_j <: x\}$. If $S_i = \emptyset$, set $SG_i = M_i$. If $S_i \neq \emptyset$, let $j = \max(S_i)$ and set $SG_i = \{x \in M_i : x_j <: x\}$. The extension L is *depth-first* if

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$x_{i+1} \in SG_i$ for all $i \geqslant 2$. We shall show that the following counting problems are NP-hard.

Greedy Linear Extensions (*GLE*):

Instance: Ordered set P, positive integer k.
Question: Does P have at least k greedy linear extensions?

Depth-first Linear Extensions (*DLE*):
Instance: Ordered set P, positive integer k.
Question: Does P have at least k depth-first linear extensions?

Finally we introduce two well-known NP-complete problems that will be needed for our proofs. Let $U = \{u_1, \ldots, u_m\}$ be a set of Boolean *variables*. A truth assignment for U is a function $f: U \to \{T, F\}$. If $u \in U$ then u and $\bar{u}$ are *literals* over U. The literal u is satisfied by f when $f(u) = T$; the literal $\bar{u}$ is satisfied by f when $f(u) = F$. A *clause* C over U is a set of literals over U. The clause C is *satisfied* by f when some literal in C is satisfied by f. A collection of clauses S is *satisfiable* when there exists a truth assignment which satisfied every clause in S. The next problem is the original NP-complete problem of Cook [1].

Satisfiability (*SAT*):
Instance: A set $U = \{u_1, \ldots, u_m\}$ of variables and a collection of clauses $S = \{C_1, \ldots, C_n\}$ over U.
Question: Is S satisfiable?

Let $G = (V, E)$ be a graph. A *vertex cover* of G is a set C of vertices such that every edge has at least one end point in C. Karp [4] showed that the next problem is NP-complete.

Vertex Cover (*VC*):
Instance: A graph $G = (V, E)$ and a positive integer $c \leqslant |V|$.
Question: Does G have a vertex cover of cardinality at most c?

2. The Proofs

THEOREM 1. *Depth-first linear extension is NP-hard.*

Proof. It suffices to show that we can transform the NP-complete problem SAT to DLE in polynomial time. Let $U = \{u_1, \ldots, u_m\}$ be a set of variables and $S = \{C_1, \ldots, C_n\}$ be a set of clauses over U, which form an arbitrary instance of SAT. We shall construct an ordered set P and an integer k such that S is satisfiable if and only if P has at least k depth-first linear extensions. For each variable u_i, there is a *truth setting* component W_i of P. A typical truth setting component is shown in Figure 1.

The truth setting components are linked together to form the *stack* by letting v_{i+1} and $\bar{v}_{i+1}$ cover w_i. The point w_m is called the *top* of the stack.

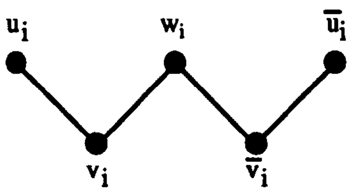

Fig. 1. Typical truth setting component W_i.

For each clause C_j, there is a *satisfaction testing* component D_j of P. A typical satisfaction testing component is shown in Figure 2. The exact value of the parameter r will be determined later in the proof.

The satisfaction testing components are called *combs*. The elements $s_j^1, \ldots, s_j^r$ form the *spine* of the comb D_j; the elements $t_j^1, \ldots, t_j^r$ are the *teeth* of the comb. The element $s_j^1 = s_j$ is called the *base* of the comb; r_j is the *root* of the comb. The satisfaction testing components are linked to the stack by letting the base of each comb cover the top of the stack; furthermore for each literal x, r_j covers x if and only if $x \in C_j$. The transformation is completed by letting $k = (r!)^n$. Figure 3 shows the transformation from $S = \{C_1, C_2\}$, where $C_1 = \{u_1, u_2, \bar{u}_4\}$ and $C_2 = \{u_2, \bar{u}_3, \bar{u}_4\}$.

The theorem follows from the next two lemmas.

LEMMA 2. *S is satisfiable if and only if P has a depth-first linear extension in which the top of the stack w_m precedes the root of every satisfaction testing component.*

Proof. First note that for each i, one of u_i and $\bar{u}_i$ must precede w_m in any depth-first linear extension L. To see this, observe that v_i and $\bar{v}_i$ are both less than w_m. If v_i precedes $\bar{v}_i$ in L, then v_i must be covered by u_i in L. Similarly, if $\bar{v}_i$ precedes v_i in L, then $\bar{v}_i$ is covered by $\bar{u}_i$ in L.

Suppose L is a depth-first linear extension in which w_m precedes every root. Let f be the truth assignment defined by $f(u_i) = T$ when $\bar{u}_i < u_i$ in L. Consider an

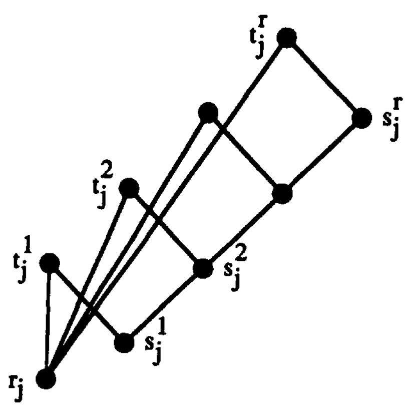

Fig. 2. Typical satisfaction testing component D_j.

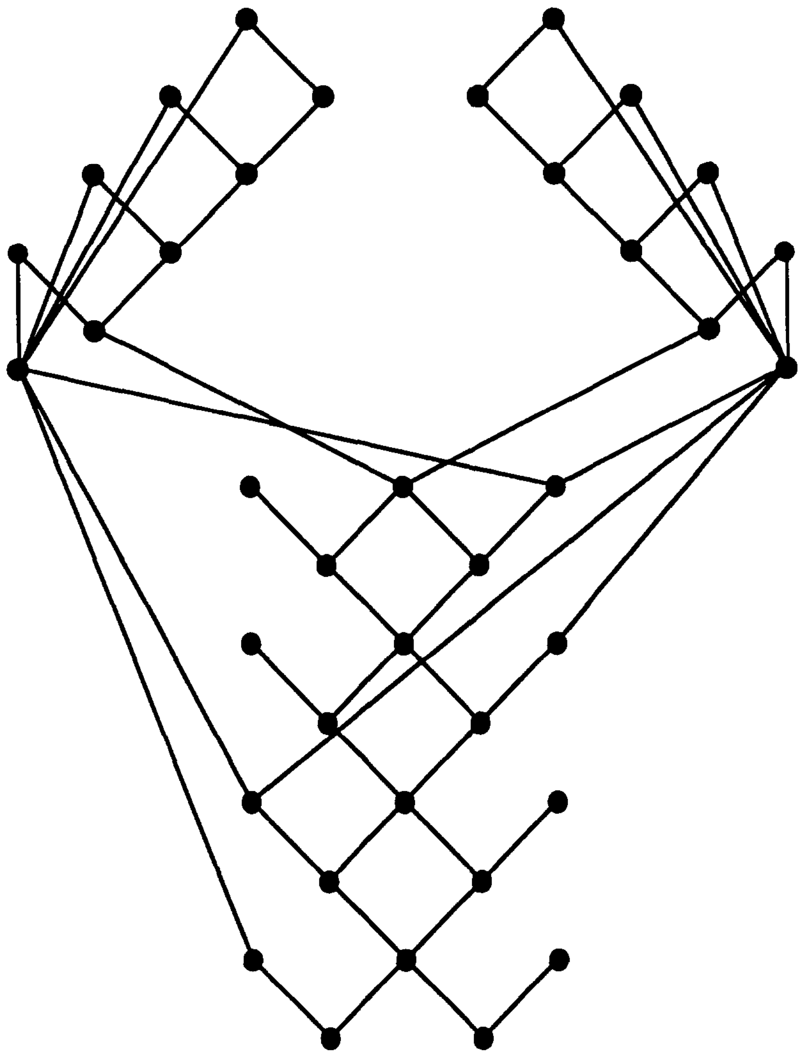

Fig. 3.

arbitrary clause C_j. Since w_m precedes r_j in L, some literal $x \in C_j$ is preceded by w_m. Thus, $\bar{x} < x$ in $L, f(x) = T$, and f satisfies C_j.

Now suppose that $f: U \to \{T, F\}$ is a truth assignment that satisfies S. Let L be the linear order defined by

$$y_1 < z_1 < \bar{y}_1 < w_1 < y_2 < z_2 < \bar{y}_2 < w_2 < \cdots < y_m < z_m < \bar{y}_m < w_m,$$

where $y_i = v_i$ and $z_i = u_i$ if $f(u_i) = F$ and $y_i = \bar{v}_i$ and $z_i = \bar{u}_i$ if $f(u_i) = T$. L is clearly an initial segment of a depth-first linear extension of the stack. Since f satisfies S, no roots must (or can) be taken before w_m. Thus, L is an initial segment of a depth-first linear extension of P in which the top of the stack precedes every root.

LEMMA 3. *P has at least $(r!)^n$ depth-first linear extensions if and only if P has a depth-first linear extension in which the top of the stack w_m precedes the root of every satisfaction testing component.*

Proof. Suppose L is a depth-first linear extension of P such that no root is less than w_m in L. Then all the spines of the combs are between w_m and all the roots and thus precede all the teeth of the combs in L. Thus, the teeth of any comb form an interval of L which is immediately preceded by the root of the comb. Moreover, any reordering of any of these interals of teeth leaves a depth-first linear extension. Since there are n intervals of r teeth, P has at least $(r!)^n$ depth-first linear extensions.

Now suppose P has no depth-first linear extension in which w_m precedes all the roots. A comb is said to be *exposed* if its root precedes the top of the stack; otherwise it is *protected.* The spine together with the teeth of an exposed comb, the spine of a protected comb, and the teeth of a protected comb all form intervals in any depth-first linear extension of P. Partition P into the stack S, the roots R, the exposed combs together with the spines of the protected combs E, and the teeth of the protected combs F. Let K_i be the set of depth-first linear extensions in which exactly i roots precede the top of the stack. We bound the size of K_i in two steps. We shall show that a depth-first linear extension of P can order

(1) S in at most 4^m ways,
(2) R in at most $n!$ ways,
(3) E in at most $n!(2^{r-1})^i$, and
(4) F in at most $(n-1)!(r!)^{n-i}$.

Then we show that there is at most one way for a depth-first linear extension to merge

(5) R with S,
(6) E with $R \cup S$, and
(7) F with $R \cup S \cup E$.

Suppose that we have shown (1)–(7). Then $|K_i| \leqslant 4^m(n!)^2 2^{(r-1)i}(n-i)!(r!)^{n-i}$. Since $i \geqslant 1$ and $2^{r-1} \leqslant r!$, this quantity is maximized when $i = 1$. Setting $s = m + n$, we get $|K_i| \leqslant 2^r(r!)^{n-1}(s!)^3/n$. Thus, summing over $i = 1, \ldots, n$, we see that P has at most $2^r(r!)^{n-1}(s!)^3$ depth-first linear extensions. If we choose $r \geqslant 3s$, then $r! \geqslant 2^r(s!)$ and so $2^r(r!)^{n-1}(s!)^3 < (r!)^n$. Thus P has fewer than $(r!)^n$ depth-first linear extensions.

It remains to show (1)–(7).

(1) The stack is made up of m truth setting components. Each can be ordered in exactly four ways by a depth-first linear extension of P:

$$v_i < u_i < \bar{v}_i < \bar{u}_i < w_i, \qquad \bar{v}_i < \bar{u}_i < v_i < u_i < w_i, \qquad v_i < u_i < \bar{v}_i < w_i < \bar{u}_i$$

or

$$v_i < u_i < \bar{v}_i < w_i < \bar{u}_i.$$

Also, there is a unique way for a depth-first linear extension to merge the truth setting components: w_i must be covered by the least element of W_{i+1} and $\{x_i : x_i \in W_i$ and w_i precedes x_i, for some $i\}$ must be ordered by $x_i < x_j$ if and only if $i > j$. Thus, there are 4^m possible orderings of the stack.

(2) Trivial.

(3) There are $n!$ ways to order the bases of the combs. Once the base of a protected comb is chosen, the spine must be taken immediately in order of the superscripts. When the base of an exposed comb is taken, both the spine and teeth must be taken immediately. There are 2^{r-1} ways to do this: Each time a nonmaximal element of the spine s_j^i is chosen there are two choices. The next element can either be s_j^{i+1} or t_j^i. If t_j^i is chosen, then the next element must be s_j^{i+1}. After choosing s_j^r the remaining teeth must be taken in reverse order of their superscripts. Since there are i exposed combs, there are $n!(2^{r-1})^i$ possible extensions.

(4) There are $n - i$ protected combs, which can be ordered in $(n - i)!$ ways. The teeth of a protected comb must be chosen immediately after the root of the comb. There are $r!$ ways to do this for each of the $n - i$ protected combs. Thus there are $(n - i)!(r!)^{n-i}$ possible extensions.

(5) Each root r_j must come between the last literal of C_j to be chosen and the next element of the stack. Thus there is at most one way to merge R with S.

(6) The base of each spine must come between the top of the stack and any other element of $R \cup S$. Thus there is at most one way to merge E with $R \cup S$.

(7) The teeth of each protected comb must come between the root of the comb and any other elements of $R \cup S \cup E$. Thus there is at most one way to merge F with $R \cup S \cup E$. □

THEOREM 4. *Greedy linear extension is NP-hard.*

Proof. We shall transform VC to GLE in polynomial time. Let the graph $G = (V, E)$ and the positive integer c form an arbitrary instance of VC. We shall construct an ordered set P and a positive integer k such that G has a vertex cover of size c if and only if P has at least k greedy linear extensions. P is constructed in two steps. First let P' be the ordered set on $V \cup E \cup \{x\}$ such that (1) for any vertex v and edge e, $v < e$ if and only if v is an end point of e and (2) x covers every vertex. For each edge e, let the *ladder* L_e be the product of a two element chain with an r element chain as shown in Figure 4. The parameter r will be determined later in the proof.

The ordered set P is formed from the ordered set P' and the ladders $\{L_e : e \in E\}$ by identifying a_e^1 with e and letting b_e^1 cover x, for all edges e. We complete the transformation by setting $k = (\lambda r + c - 1)!/(r!)^\lambda$, where $\lambda = |E| - c + 1$. Figure 5 shows the ordered set which corresponds to P_3, the path on three vertices, when $r = 4$.

The theorem follows from the next two lemmas.

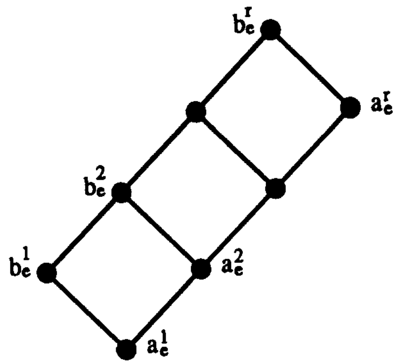

Fig. 4. The ladder L_e.

LEMMA 5. *G has a vertex cover of size at most c if and only if P has a greedy linear extension which puts at least λ edges over x.*

Proof. Suppose C is a vertex cover of size at most c. Form a greedy linear extension of P by first choosing the vertices of $V - C$. This is possible since all edges cover a vertex in C, which has not yet been chosen. Choose the remaining points of the greedy linear extension by avoiding edges wherever possible. After choosing a vertex from C we may be forced to then take an edge, but we will never be forced to take more than one edge. After choosing the last vertex from C, we can avoid taking an edge by choosing x. This leaves at least λ edges to be chosen after x.

Now suppose L is a greedy linear extension of P, which puts at least λ edges over x. Let $C = \{v \in V : v <: y \text{ in } L, \text{ for some } y \in E \cup \{x\}\}$. Clearly $|C| \leqslant c$. To see that C is a vertex cover, consider an edge $e = vw$. Suppose $v < w$ in L. When w is chosen

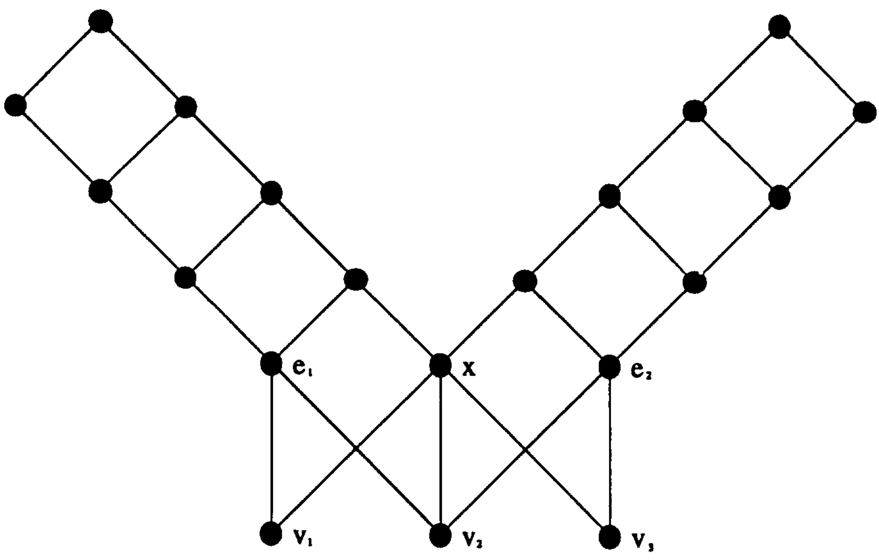

Fig. 5.

e is minimal among the remaining elements and covers w. Thus e covers w in L unless some other edge or x covers w in L. In either case $w \in C$. □

LEMMA 6. *P has at least k greedy linear extensions if and only if P has a greedy linear extension which puts at least λ edges over x.*

Proof. The chain $\{b_e^1, \ldots, b_e^r\}$ is called the *upper* chain of L_e; the chain $\{a_e^1, \ldots, a_e^r\}$ is called the *lower* chain of L_e. Let $D = V \cup E \cup \{x\}$, $d = |D|$, and M be a linear extension of D. An edge e is said to be *exposed* in M if e precedes x in M; otherwise e is *protected* in M.

Suppose P has a greedy linear extension M, which puts at least λ edges above x. Let $R = \{a_e^i : e \text{ is protected in } M\} \cup \{b_e^1 : e \text{ is exposed in } M\}$. Then $|R| = \lambda r + c - 1$ and there are $k = (\lambda r + c - 1)!/(r!)^\lambda$ linear extensions of R. Each linear extension L of R is contained in a greedy linear extension of P formed as follows: Let M_x be M restricted to $\{y : y \leqslant x \text{ in } M\}$ and L^+ be L with b_e^i inserted immediately after a_e^i, if e is a protected edge, and the rest of the upper chain of L_e inserted immediately after b_e^1, if e is an exposed edge. Then $M_x + L^+$ is the desired greedy linear extension of P. We conclude that P has at least k greedy linear extensions.

Suppose no greedy linear extension of P puts all but $c - 1$ edges above x. Fix a greedy linear extension M of P. Let $H = \{a_e^i : e \text{ is exposed in } M\}$ and $K = \cup\{L_e : e \text{ is protected in } M\} \cup \{b_e^i : e \text{ is exposed in } M\}$. We shall show that greedy linear extensions of P, which agree with M on D, can order

(1) $D \cup H$ in at most one way and
(2) K in at most $r^{\lambda-1}[r(\lambda-1)+c]!/(r!)^{\lambda-1}$ ways.

Suppose we have shown (1) and (2). Then the number of greedy linear extensions of P is at most $d!r^{\lambda-1}[r(\lambda-1)+c]!/(r!)^{\lambda-1}$, which is less than $k = (\lambda r + c - 1)!/(r!)^\lambda$, when we set $r = d\ln(d)$.

It only remains to show (1) and (2).

(1) If e is exposed in M, then the lower chain of L_e must be chosen immediately after e in any greedy linear extension of P.

(2) Note that if for some edge e, a_e^i is covered by a_e^{i+1} in a greedy linear extension L of P, then for all $j \geqslant i$, a_e^j is covered by a_e^{j+1} in L. Thus, if e is a protected edge, then L_e can be ordered in r ways, each with at most r jumps, by greedy linear extensions of P. If e is an exposed edge, the upper chain of L_e must be chosen immediately after b_e^1 in any greedy linear extension of P. There are at most $\lambda - 1$ protected edges in P. Thus, there are at most $r^{\lambda-1}$ ways to order the parts of ladders, which are contained in K, and $[r(\lambda-1)+c]!/(r!)^{\lambda-1}$ ways to merge these parts in a greedy linear extension of P. The result follows. □

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