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THE MAXIMUM NUMBER OF EDGES IN $2K_2$-FREE GRAPHS OF BOUNDED DEGREE

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A graph is $2K_2$-free if it does not contain an independent pair of edges as an induced subgraph. We show that if G is $2K_2$-free and has maximum degree $\Delta(G) = D$, then G has at most $5D^2/4$ edges if D is even. If D is odd, this bound can be improved to $(5D^2 - 2D + 1)/4$. The extremal graphs are unique.

1. Introduction

We call a graph $2K_2$-free if it is connected and does not contain two independent edges as an induced subgraph. The assumption of connectedness in this definition only serves to eliminate isolated vertices. Wagon [6] proved that $\chi(G) \leq \omega(G)[\omega(G)+1]/2$ if G is $2K_2$-free where $\chi(G)$ and $\omega(G)$ denote respectively the chromatic number and maximum clique size of G. Further properties of $2K_2$-free graphs have been studied in [1, 3, 4 and 5].

$2K_2$-free graphs also arise in the theory of perfect graphs. For example, split graphs and threshold graphs are $2K_2$-free (see [2]). On the other hand, the strong perfect graph conjecture is open for the class of $2K_2$-free graphs.

In this paper we solve the following extremal problem posed by Bermond et al. in [7] and also by Nešetřil and Erdös: What is the maximum number of edges in a $2K_2$-free graph with maximum degree D? Our principal result asserts that the extremal graph is unique for all D and can be obtained from the five-cycle by multiplying its vertices. The extremal problem solved here is a special case of a more general conjecture of Erdös and Nešetřil which can be viewed as a variation on Vizing's Theorem: Two edges are said to be strongly independent if there is no edge incident to both edges. They conjecture that if $\Delta(G) = D$, the edge set of G can be partitioned into at most $5D^2/4$ color classes in such a way that any two

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edges in the same color class are strongly independent. It is not difficult to see that $2D^2$ colors suffices. Our result in this paper provides a lower bound of $5D^2/4$ by showing certain graphs require $5D^2/4$ colors.

The proof of our result is based on some structural properties of $2K_2$-free graphs. The most general of these properties are collected in Section 2. The special properties concerning $2K_2$-free graphs with clique size 3 or 4 are established as claims within the proof of the theorem in Section 3. Some of the proof techniques we employ are similar to those used in [5].

Throughout the paper, $V(G)$ and $E(G)$ denote the vertex set and edge set of the graph G. For a vertex $x \in V(G)$, $N(x)$ is the set of neighbors of x. For disjoint subsets A, B of $V(G)$ we let $[A, B]$ denote the bipartite subgraph of G whose vertex set is $A \cup B$ and whose edge set consists of those edges in G with one endpoint in A and the other in B. For a vertex $x \in V(G)$ and a positive integer n, we say H is obtained from G by multiplying x by n when H is formed by replacing the vertex x by a stable (independent) set of n vertices each having the same neighbors as x.

2. Structural properties of $2K_2$-free graphs

We will first prove several structural properties of $2K_2$-free graphs which turn out to be very useful in the proof of the main theorem.

Theorem 1. *Let G be a $2K_2$-free graph, A be a stable set of G, and $B = V(G) - A$. There exist $x \in B$ such that $N(x)$ meets all edges of $[A, B]$.*

Proof. Consider the bipartite graph G' determined by the edges of $[A, B]$. We choose $x \in B$ such that x has maximum degree in G'. Consider $N(x)$ in G and set $A' = N(x) \cap A$, $B' = N(x) \cap B$. Assume that x does not satisfy the conclusion of our theorem, i.e. assume that $N(x) \cap \{p, q\} = 0$ for some $pq \in E(G)$, $p \in A$, $q \in B$. For any $\tau \in A$;, $\tau p \notin E(G)$ because A is stable, $xp, xq \notin E(G)$ by the definition of A' and B'. Since G is $2K_2$-free, $\tau q \in E(G)$, and it follows in G' that the degree of q is larger than the degree of x in G', contradicting the choice of x. □

Corollary. *If G is a bipartite $2K_2$-free graph then both color classes of G contain vertices adjacent to all vertices of the other color class of G.*

Theorem 2. *Assume that G is $2K_2$-free, $\omega(G) = 2$ and G is not bipartite. Then G can be obtained from a five-cycle by vertex multiplication.*

Proof. Since G is $2K_2$-free, minimum-length odd cycles of G must be of length 5. If x_1, x_2, x_3, x_4, x_5 are the vertices of a five-cycle C of G, let A_i denote the set of vertices in G adjacent to x_i and x_{i+2} for each $i = 1, 2, \ldots, 5$ (cyclically). Clearly the sets A_i are stable and form a partition of $V(G)$. From this, it follows easily that G can be obtained from C by multiplying x_i by $|A_i|$. □

For a subset $X \subset V(G)$, we let $\mathrm{Dom}(X)$ denote the set of vertices dominated by X, i.e. $\mathrm{Dom}(X) = X \cup \{y \in V(G);$ there exists $x \in X$ such that $xy \in E(G)\}$. The set X is said to be dominating if $\mathrm{Dom}(X) = V(G)$. A *dominating clique* of a graph G is a dominating set which induces a complete subgraph in G. The following result is a variant of a theorem of El-Zahar and Erdös [1].

Theorem 3. *If G is $2K_2$-free and $\omega(G) \geq 3$, then G has a dominating clique of size $\omega(G)$.*

Proof. Let $\omega(G) = p \geq 3$. Among all the p-element cliques in G, choose one, say $K = \{x_1, x_2, \ldots, x_p\}$ so that $t = |V(G) - \mathrm{Dom}(K)|$ is minimum. If $t = 0$, then K is dominating, so we may assume $t > 0$. Let $Z = V(G) - \mathrm{Dom}(K)$. Since $p \geq 2$, Z is a stable set. For each $i = 1, 2, \ldots, p$, let $Y_i = \{y \in \mathrm{Dom}(K): yx_j \in E(G)$ if and only if $i = j\}$. Since $p \geq 3$, each Y_i is a stable set.

Choose an arbitrary element $z_0 \in Z$ and let $y_0 \in \mathrm{Dom}(K)$ be any neighbor of z_0. Since G is $2K_2$-free and p is maximal, there is a unique integer $i \leq p$ so that $y_0x_j \in E(G)$ if and only if $i \neq j$. Therefore $K' = (K - \{x_i\}) \cup \{y_0\}$ is a clique of size p. Furthermore, any vertex dominated by K is dominated by K' except possibly those vertices in the set $Y_i' = \{y \in Y_i : y_0y \notin E(G)\}$. Since $z_0 \in \mathrm{Dom}(K')$, the minimality of t requires that $Y_i' \neq \emptyset$. Let $y_1 \in Y_i'$. Then the edges z_0y_0 and x_iy_1 force $z_0y_1 \in E(G)$. Choose distinct $j, k \in \{1, 2, \ldots, p\} - \{i\}$. Then z_0y_1 and x_jx_k are independent edges. The contradiction completes the proof. □

3. The extremal result

The main result of this section is the determination of the maximum number of edges in a $2K_2$-free graph with a given maximum degree. It is convenient to introduce the notation $C_5(D)$ for the following graph. If D is even, then $C_5(D)$ denotes the graph obtained from the five cycle C_5 by multplying each vertex of C_5 by $D/2$. If D is odd then $C_5(D)$ denotes the graph obtained from C_5 by multiplying two consecutive vertices by $(D+1)/2$ and the other three vertices by $(D-1)/2$. Let $f(D) = |E(G)|$ denote the number of edges of $C_5(D)$. Obviously $f(D) = 5D^2/4$ if D is even and $f(D) = (5D^2 - 2D + 1)/4$ if D is odd.

Theorem 4. *Let $D \geq 2$. If G is $2K_2$-free and the maximum degree of G is at most D, then $|E(G)| \leq f(D)$. Equality holds if and only if G is isomorphic to $C_5(D)$.*

Actually, we will prove a more technical result from which Theorem 4 is readily extracted.

Theorem 5. *Let $D \geqslant 2$ and suppose that G is a $2K_2$-free graph with maximum degree at most D.*

(i) *If G is bipartite, then $|E(G)| \leqslant D^2$. Equality holds if and only if G is the complete bipartite graph $K_{D,D}$.*
(ii) *If $\omega(G) = 2$ and G is not bipartite, then $|E(G)| \leqslant f(D)$. Equality holds if and only if G is isomorphic to $C_5(D)$.*
(iii) *If $\omega(G) \geqslant 5$ then $|E(G)| \leqslant (5D^2 - 5D - 20)/4 < f(D)$.*
(iv) *If $\omega(G) = 4$ then $|E(G)| \leqslant (5D^2 - 3D - 10)/4 < f(D)$.*
(v) *If $\omega(G) = 3$ then $|E(G)| < f(D)$.*

Proof of (i). The statement follows immediately from the Corollary to Theorem 1. □

Proof of (ii). From Theorem 2, we know that G is obtained from C_5 by vertex multiplications. Assume that C_5 contains vertices x_1, x_2, x_3, x_4, x_5 and G is obtained from C_5 by multiplying each x_i by a_i. It is elementary to show that $\sum_{i=1}^{5} a_i a_{i+1} \leqslant f(D)$ under the condition $a_i + a_{i+2} \leqslant D$ (subscript arithmetic is taken modulo 5) and that equality holds only for $C_5(D)$. □

We will find it convenient to introduce some notation before proceeding with the proofs of the remaining parts. If $\omega(G) = p \geqslant 3$, then we can choose a dominating clique $K = \{x_1, x_2, \ldots, x_p\}$ in G using Theorem 3. Then let $Y = V(G) - K$. If S is a nonempty subset of $\{1, 2, \ldots, p\}$, we denote by $A(S)$ the set of vertices defined by $A(S) = \{y \in Y : yx_i \in E(G)$ if and only if $i \in S\}$. The family $\{A(S) : S \subseteq \{1, 2, \ldots, p\}, S \neq \emptyset\}$ is a partition of Y. For a set $S = \{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, p\}$, we will also write $A(i_1, i_2, \ldots, i_k)$ for $A(S)$.

When $y_1, y_2 \in Y$ and $y_1y_2 \in E(G)$, we define the *weight* of the edge y_1y_2, denoted $w(y_1y_2)$, as $|N(y_1) \cap K| + |N(y_2) \cap K|$. The following claim follows immediately from the fact that G is $2K_2$-free.

Claim 0. If $y_1, y_2 \in Y$ and $y_1y_2 \in E(G)$, then $w(y_1y_2) \geqslant p - 1$.

Proof of (iii). There are at most $\binom{p}{2} + p(D - p + 1)$ edges incident to the vertices of K. Moreover, since every $x_i \in V(K)$ has at most $D - p + 1$ neighbors in Y, for the edges contained in Y, we obtain

$$\sum_{e \in Y} w(e) \leqslant p(D - p + 1)(D - 1). \qquad (*)$$

By Claim 1, $w(e) \geqslant p - 1$ for all $e \in Y$, so that

$$|E(G)| \leqslant \binom{p}{2} + p(D - p + 1) + \frac{p}{p-1}(D - p + 1)(D - 1)$$
$$= \frac{p}{p-1}D^2 - \frac{p}{p-1}D - \frac{p(p-3)}{2}.$$

For $p \geqslant 5$, this upper bound on the number of edges in G is a decreasing function of p, which completes the proof of (iii). □

Proof of (iv). If $p = 4$, inequality (*) above implies $|E(G)| \leqslant 4D - 6 + (D - 1)(D - 3) + \frac{1}{4}|E_3| = \frac{1}{4}|E_3| + d^2 - 3$ where E_3 is the set of edges $e \subset Y$ having weight three. Let A^j denote the subset of Y constiting of those vertices with exactly j neighbors in K. Then if e is an edge in E_3, then one end point of e is in A^1 and the other is in A^2. Furthermore the set A^1 is easily seen to be a stable set. By applying Theorem 1 to the subgraph of G induced by $A^1 \cup A^2$, there exists a vertex $y \in A^2$ so that $N(y)$ meets all edges in $E_3 = [A^1, A^2]$. Now y has at most $D - 2$ neighbors in Y and each of these meets at most $D - 1$ edges in E_3. We conclude that $|E_3| \leqslant (D - 1)(D - 2)$. Thus $E(G) \leqslant (5D^2 - 3D - 10)/4$. □

Proof of (v). The proof for this case is somewhat complicated. The argument is by contradiction. We assume that $|E(G)| \geqslant f(D)$. Then $|V(G)| \geqslant 2f(D)/D$. Since $p = 3$, we know that $Y = A(12) \cup A(13) \cup A(23) \cup A(1) \cup A(2) \cup A(3)$. We will establish a series of claims which yield the proof.

Claim 1. $|Y| > (5D - 8)/2$.

Proof. Suppose not. If D is even, then $|Y| \leqslant (5D - 8)/2$ implies

$$|E(G)| \leqslant |Y|\,(D - 1)/2 + 3 + 3(D - 2) \leqslant (5D - 8)(D - 1)/4 + 3D - 3$$
$$= (5D^2 - D - 4)/4 < 5D^2/4 = f(D).$$

If D is odd, then $|Y| \leqslant (5D - 9)/2$, so $|E(G)| \leqslant (5D^2 - 2D - 3)/4 < f(D)$. □

Claim 2. $|A(1)| > |A(23)| + D/2$, $|A(2)| > |A(13)| + D/2$ and $|A(3)| > |A(12)| + D/2$.

Proof. $|Y| = |N(x_2) \cap Y| + |N(x_3) \cap Y| + |A(1)| - |A(23)| \leqslant 2(D - 2) + \|A(1)| - |A(23)|$. Since $|Y| > (5D - 8)/2$, we conclude $|A(1)| > |A(23)| + D/2$. The other inequalities follow by symmetry. □

Let $\lambda_1 = |A(1)| + |A(2)| + |A(3)|$ and $\lambda_2 = |A(12)| + |A(13)| + |A(23)|$. Then $|Y| = \lambda_1 + \lambda_2$ and $3D - 6 \geqslant \lambda_1 + \lambda_2$.

Claim 3. $\lambda_2 < (D - 4)/2$.

Proof. Suppose $\lambda_2 \geqslant (D - 4)/2$. Then $3D - 6 \geqslant \lambda_1 + 2\lambda_2 = \lambda_1 + \lambda_2 + \lambda_2 \geqslant |Y| + (D - 4)/2$. Thus $|Y| \leqslant (5D - 8)/2$, contradicting Claim 1. □

Claim 4. $A(1) \cup A(2) \cup A(3)$ is not a stable set.

Proof. If $A(1) \cup A(2) \cup A(3)$ is a stable set, then $|E(G)| \leqslant 3D - 3 + \lambda_2(D-2) < 3D - 3 + (D-4)(D-2)/2 \leqslant f(D)$. □

Claim 5. $A(1) \cup A(2)$, $A(2) \cup A(3)$, and $A(1) \cup A(3)$ are not stable sets.

Proof. Suppose $A(1) \cup A(2)$ is a stable set. By Claim 4, we know there is an edge in $A(1) \cup A(2) \cup A(3)$, so we may assume there is an edge xz where $x \in A(1)$ and $z \in A(3)$. Now let y be an arbitrary vertex in $A(2)$. The edges xz and x_2y show $yz \in E(G)$. Now let $x' \in A(1)$. Then the edges $x'x_1$ and zy show $x'z \in E(G)$. Thus z is adjacent to every vertex in $A(1) \cup A(2)$. This is impossible since $|A(1) \cup A(2)| > D$ by Claim 2. □

Claim 6. Let i, j be distinct integers from $\{1, 2, 3\}$. Then one of the following statements holds.

(i) There exists $x \in A(i)$ with $xy \in E(G)$ for every $y \in A(j)$.

(ii) There exists $y \in A(j)$ with $xy \notin E(G)$ for every $x \in A(i)$.

Proof. Assume statement (ii) does not hold. Choose $x \in A(i)$ so that $|N(x) \cap A(j)|$ is maximum. If x has a nonneighbor $y \in A(j)$, choose a neighbor x^* of y from $A(i)$. Then x^* has more neighbors in $A(j)$ then x. □

Let i, j be distinct elements of $\{1, 2, 3\}$. We say $A(i)$ and $A(j)$ are *linked* if there exists an element $x \in A(i)$ adjacent to all points in $A(j)$ and an element $y \in A(j)$ adjacent to all points in $A(i)$.

Claim 7. There exist distinct integers $i, j \in \{1, 2, 3\}$ so that $A(i)$ and $A(j)$ are linked.

Proof. If $A(1)$ and $A(2)$ are not linked, we may assume without loss of generality that there exists $y_0 \in A(2)$ so that $xy_0 \notin E(G)$ for every $x \in A(1)$. By Claim 5, there exists an edge x_0z_0 between $A(1)$ and $A(3)$. Thus $z_0y_0 \in E(G)$. Therefore $z_0x \in E(G)$ for every $x \in A(1)$. By Claim 2 we can choose $y_1 \in A(2)$ so that $z_0y_1 \notin E(G)$. Then $y_1x \in E(G)$ for every $x \in A(1)$. If $A(1)$ and $A(3)$ are not linked, then there exists $z_1 \in A(3)$ with $z_1x \notin E(G)$ for every $x \in A(1)$. The edge x_0y_1 shows $y_1z_1 \in E(G)$. The edges y_0z_0 and y_1z_1 require $y_0z_1 \in E(G)$. But this implies that y_0z_1 and x_1x_0 are independent. □

We are now ready to obtain the final contradiction. By Claim 7, we may assume that $A(1)$ and $A(2)$ are linked. We choose $a_0 \in A(1)$, $b_0 \in A(2)$ so that a_0b and ab_0 are edges in G for every $b \in A(2)$ and every $a \in A(1)$. Now every vertex of Y is adjacent to either a_0 or b_0 except possibly those points in $A(12)$. This implies that $|Y| \leqslant 2(D-1) + |A(12)|$. The inequality $|Y| > (5D-8)/2$ then requires $|A(12)| > (D-4)/2$. This contradicts Claim 3 since $|A(12)| \leq \lambda_2 < (D-4)/2$. With this observation, the proof of our theorem is complete. □

4. Concluding remarks

The problem we dealt with here can be viewed as a variation of Turan's Theorem. Namely, for a given forbidden graph H, it is of interest to determine the maximum number of edges in a graph G on n vertices which does not contain H as an induced subgraph subject to certain degree constraints on G. Turan's Theorem considers the case of H as cliques. In this paper we investigate the case of H as $2K_2$. To consider the corresponding problem for a general class of H, it is essential to establish a clear understanding of the structural properties for graphs which does not contain H as an induced subgraph. This is indeed a fundamental problem in graph theory where more research is needed.

Another direction is along the line of the general conjecture of Erdős and Nešetřil of coloring the edges of a graph such that two monochromatic edges are strongly independent. Such an edge coloring will be called a strong edge coloring. Their conjecture that $5D^2/4$ color suffices for graphs of maximum degree D is an intriguing problem. Clearly more ideas are required to attack this problem successfully. The problem of strong edge-coloring for general graphs opens up a wide range of problems of edge coloring which we will not discuss here.

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