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A Note on Removable Pairs

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ABSTRACT

A long standing conjecture in the dimension theory for finite ordered sets asserts that every ordered set (of at least three points) contains a pair whose removal decreases the dimension at most one. Two stronger conjectures have been made:

- (1) *If (x, y) is a critical pair, then $\dim(P) \leq 1 + \dim(P - \{x, y\})$.*
- (2) *For every $x \in P$, there exists $y \in P - \{x\}$ so that $\dim(P) \leq 1 + \dim(P - \{x, y\})$.*

K. Reuter has disproved conjecture 1 by constructing a four-dimensional poset P containing a critical pair (x, y) so that $\dim(P - \{x, y\}) = 2$. In this note, we construct for every $n \geq 5$ an n -dimensional poset P_n containing a critical pair (x, y) so that $\dim(P_n - \{x, y\}) = n - 2$. Point y is a maximal point of P_n .

1. Preliminaries

Recall that the *dimension* of a finite ordered set P is the least positive integer t so that there exist t linear extensions L_1, L_2, \dots, L_t so that $P = L_1 \cap L_2 \cap \dots \cap L_t$. An incomparable pair (x, y) is called a *critical pair* if any point less than x is less than y and any point greater than y is greater than x . The dimension of P is the least t for which there exist t linear extensions of P so that for every critical pair (x, y) , there is at least one i for which $y < x$ in L_i . We refer the reader to the survey article [3] by D. Kelly and W.T. Trotter and the chapters [6], [7] by Trotter for additional background information on dimension theory.

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2. Removable Pairs

The following conjecture is one of the best known open problems in dimension theory and is a featured problem in ORDER. We believe the first reference to the conjecture is [1].

Conjecture 0 If P is an ordered set having at least three points, then P contains a distinct pair (x, y) so that $\dim(P) \leq 1 + \dim(P - \{x, y\})$.

A pair $x, y \in P$ for which $\dim(P) \leq 1 + \dim(P - \{x, y\})$ is called a 1-removable pair, so that Conjecture 0 asserts that every poset contains a 1-removable pair.

The first reference to the following conjecture is apparently [5].

Conjecture 1 Every critical pair is 1-removable.

In [2], D. Kelly made the following conjecture which is also stronger than

Conjecture 0.

Conjecture 2 For every $x \in P$, there is a point $y \in P - \{x\}$ so that x, y is a

1-removable pair.

K. Reuter [4] has disproved Conjecture 1 by constructing the ordered set shown in

Figure 1. This ordered set P has dimension 4, (x, y) is a critical pair, and $\dim(P - \{x, y\}) = 2$. Note that y is a maximal point.

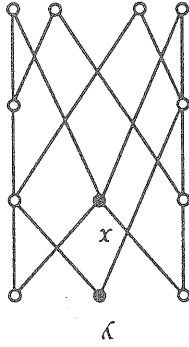
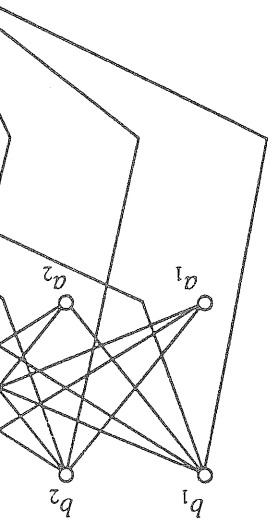


Figure 1

The purpose of this note is to show that Reuter's example is not an isolated phenomenon. To accomplish this, we will establish the following result.

We first show that d



Theorem For every n
a critical pair (x, y)
1-removable, i.e., $\dim(P)$
Proof For $n = 4$, we h
set of P_n contains $4n -$
 $\{c_i : 1 \leq i \leq n - 2\} \cup$
 $j \leq n - 2$ and $i \neq j$, we h
 $1 \leq i \leq n - 2$, we have
 $z < d_i$. We also have w
 $n = 5$.

Theorem For every $n \geq 4$, there exists an n -dimensional ordered set P_n containing a critical pair (x, y) so that y is a maximal element in P_n , but (x, y) is not 1 -removable, i.e., $\dim(P - \{x, y\}) = n - 2$.

Proof For $n = 4$, we have Reuter's example shown in Figure 1. For $n \geq 5$, the point set of P_n contains $4n - 4$ points labelled $\{a_i : 1 \leq i \leq n - 2\} \cup \{b_i : 1 \leq i \leq n - 2\} \cup \{c_i : 1 \leq i \leq n - 2\} \cup \{d_i : 1 \leq i \leq n - 2\} \cup \{x, y, z, w\}$. For all i, j with $1 \leq i, j \leq n - 2$ and $i \neq j$, we have the cover relations $a_i < b_j$ and $c_i < d_j$. For each i with $1 \leq i \leq n - 2$, we have $a_i < y, c_i < y, c_i < x, z < b_i, w < b_i, w < d_i, x < b_i$, and $z < d_i$. We also have $w < y$. We illustrate this definition with a diagram for P_n when $n = 5$.

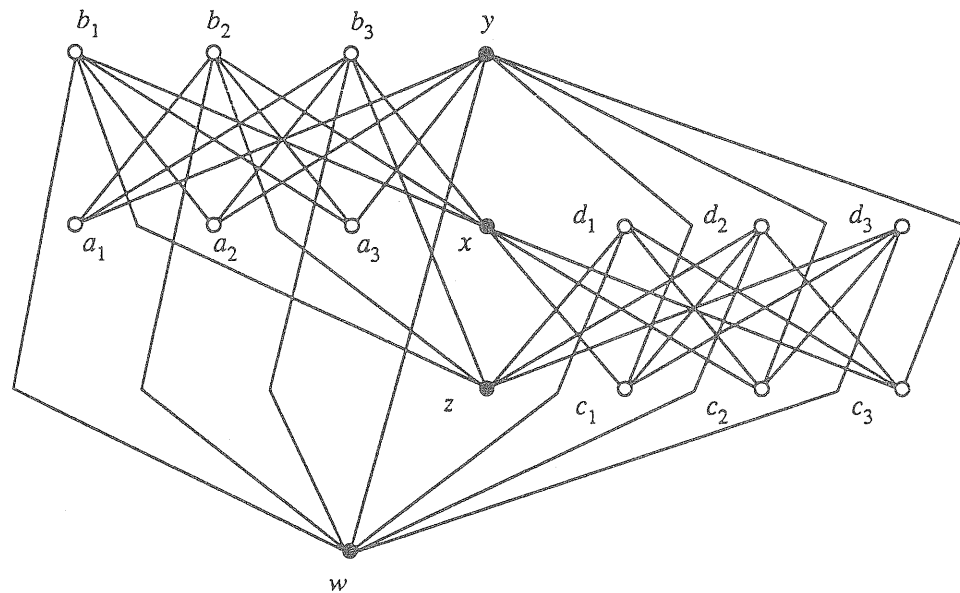


Figure 2

We first show that $\dim(P_n) \geq n$. To the contrary, suppose $\dim(P_n) \leq n - 1$, and let L_1, L_2, \dots, L_{n-1} be linear extensions whose intersection is P_n . Without loss of generality, we may assume that $b_i < a_i$ in L_i for $i = 1, 2, \dots, n - 2$. Thus we must have $x > y$ in L_{n-1} and $z > y$ in L_{n-1} . However, this implies that for each $i = 1, 2, \dots, n - 2$, there exists a unique $j_i \in \{1, 2, \dots, n - 2\}$ so that $c_i > d_i$ in L_{j_i} . Hence

