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Interval Orders and Shift Graphs

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ABSTRACT

A finite partially ordered set (poset) $\mathbf{P} = (X, P)$ is called an interval order if there is a $1-1$ function which assigns to each element $x \in X$ a closed interval $[a_x, b_x]$ of the real line $\mathbb{R}$ so that $x < y$ in P if and only if $b_x < a_y$ in $\mathbb{R}$. For a poset (X, P), height(X, P) is the maximum number of points in a chain, while $\dim(X, P)$ is the dimension of (X, P), the minimum number of linear orders on X whose intersection is the partial order P. In 1972, I. Rabinovitch proved that the function $f(n) = \max\{\dim(X, P) :\ \text{height}(X, P) \leq n,\ P \text{ is an interval order}\}$ is defined and satisfies $f(n) \leq \lceil n+1 \rceil$. In this paper, we show that $f(n) = \lg\lg n + (1/2 + \ o(1))(\lg\lg\lg n)$. The proof techniques include establishing links between the dimension of interval orders, the chromatic number of shift graphs, and the classical problem of counting the number of antichains in the subset lattice.

1. Introduction

We consider a *partially ordered set* $\mathbf{P}$ (also called a *poset* or *ordered set*) as a pair (X, P) where X is a set (always finite in this paper) and P is a

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reflexive, antisymmetric and transitive binary relation on X. The *dimension* of a poset $\mathbf{P} = (X, P)$, denoted by $\dim(\mathbf{P})$ or $\dim(X, P)$, is the least t so that the partial order P is the intersection of t linear orders on X. We refer the reader to the monograph [14] and the survey articles [6], [11], [12] for additional material on dimension theory for posets.

The *height* of a poset $\mathbf{P} = (X, P)$, denoted by $\text{height}(X, P)$, is the largest n so that (X, P) contains a chain of n points. A poset of height 1 is an antichain and has dimension at most 2. A poset of height 2 can have arbitrarily large dimension. Examples are given in [7].

There is an important class of posets for which the dimension is bounded in terms of height. A poset $\mathbf{P} = (X, P)$ is an *interval order* if there is a $1-1$ function assigning to each element $x \in X$ a closed[†] interval $[a_x, b_x]$ of the real line $\mathbb{R}$ so that $x < y$ in P if and only if $b_x < a_y$ in $\mathbb{R}$. The following well-known theorem of P. Fishburn [4] gives a forbidden subposet characterization of interval orders.

Theorem 1.1. (P. Fishburn). *A poset $\mathbf{P} = (X, P)$ is an interval order if and only if X does not contain four points x_1, x_2, y_1, y_2 with $x_1 < x_2$, $y_1 < y_2$, $x_1 \not< y_2$, and $y_1 \not< x_2$ in P.* ∎

In 1972, I. Rabinovitch [9] proved that the function $f(n) = \max\{\dim(X, P) : (X, P)$ is an interval order and height $(X, P) \leq n\}$ is well defined and satisfies $f(n) \leq \lceil n + 1 \rceil$. A logarithmic upper bound on $f(n)$ was proved by Bogart, Rabinovitch and Trotter in [1]. The primary goal of this paper will be to prove a surprisingly tight asymptotic formula for $f(n)$. Hereafter, all logarithms are base 2.

Theorem 1.2. *The maximum dimension $f(n)$ of an interval order of height n satisfies:*

$$f(n) = \lg\lg n + \left(\frac{1}{2} + o(1)\right) \lg\lg\lg n. \quad ∎$$

In the next section of the paper, we establish combinatorial connections between the dimension of an interval order, the chromatic number of shift graphs, and the number of antichains in the lattice of subsets of a finite set. In order to exploit these connections, we will also extend the concept of shift graphs to hypergraphs.

[†] The essential property is that each $x \in X$ be represented by a connected subset of $\mathbb{R}$. In this paper, we follow the most widely used convention and require that the connected set be a nondegenerate closed interval.

2. Shift graphs and antichains in subset lattices

In this paper, we use $[n]$ to denote the finite set $\{1, 2, \dots, n\}$. For a set X and a nonnegative integer m, $\binom{X}{m}$ denotes the set of all m-element subsets of X. Whenever $S = \{x_1, x_2, \dots, x_t\}$ is a set of integers, we assume that the elements of S have been labelled so that $x_i < x_j$ whenever $i < j$.

For integers n, k with $n \geq k + 1$, Erdős and Hajnal [2] (see also [5]) defined the *shift graph* $\mathbf{G}(n, k)$ as a graph whose vertex set is $\binom{[n]}{k}$ and whose edge set consists of all pairs of the form $\{\{x_1, x_2, \dots, x_k\}, \{x_2, x_3, \dots, x_{k+1}\}\}$ where $S = \{x_1, x_2, \dots, x_{k+1}\} \in \binom{[n]}{k+1}$. The shift graph $\mathbf{G}(n, 1)$ is a complete graph on n vertices, and of course its chromatic number is n. It is an easy exercise to show that $\chi(\mathbf{G}(n, 2)) = \lceil \lg n \rceil$. For larger k, Erdős and Hajnal [2] gave the following asymptotic formula.

Theorem 2.1. (Erdős and Hajnal) *For fixed* $k \geq 3$, $\chi(\mathbf{G}(n, k)) = (1 + o(1)) \lg \lg \dots \lg n$, *where the logarithm function is iterated* $k - 1$ *times.* ■

In this paper, we are concerned primarily with the family $\{\mathbf{G}(n, 3) : n \geq 4\}$. These graphs are also called *double shift* graphs. For double shift graphs, we can provide a more accurate estimate on the chromatic number than is provided by (2.1). For an integer $t \geq 1$, let $\mathbf{2}^t$ denote the poset consisting of all subsets of $[t]$ partially ordered by inclusion. The poset $\mathbf{2}^t$ is a distributive lattice having dimension t, height $t + 1$ and width $\binom{t}{\lfloor \frac{t}{2} \rfloor}$.

The following result is part of the folklore of this subject, but to the best of our knowledge, no one has published a proof. We do so now because it directly motivates arguments which follow.

Theorem 2.2. *For each integer* $n \geq 4$, *the chromatic number of the double shift graph* $\mathbf{G}(n, 3)$ *is the least* t *for which there are at least* n *antichains in* $\mathbf{2}^t$, *the lattice of all subsets of* $[t]$.

Proof. Suppose first that $\chi(\mathbf{G}(n, 3)) = t$. We show that $\mathbf{2}^t$ has at least n antichains. Let $\psi : \binom{[n]}{3} \to [t]$ be a good coloring of $\mathbf{G}(n, 3)$. For each $A = \{i_1, i_2\} \in \binom{[n]}{2}$, let $S_A = \{\alpha \in [t] :$ there exists $i_3 \in [n]$ with $i_3 > i_2$ and $\psi(\{i_1, i_2, i_3\}) = \alpha\}$. For each $i_1 \in [n]$, let $\mathcal{C}_{i_1} = \{S_A :$ there exists $i_2 \in [n]$ with $i_2 > i_1$ and $A = \{i_1, i_2\}\}$. Partial order each $\mathcal{C}_{i_1}$ by inclusion, and let $\mathcal{M}_{i_1}$ be the set of maximal elements. Note that $\mathcal{M}_{i_1}$ is an antichain in the lattice $\mathbf{2}^t$.

Next, we show that the n antichains in $\{\mathcal{M}_{i_1} : i_1 \in [n]\}$ are distinct. Suppose to the contrary that $A = \{i_1, i_2\} \in \binom{[n]}{2}$ and $\mathcal{M}_{i_1} = \mathcal{M}_{i_2}$. Then $S_A \in \mathcal{C}_{i_1}$, so there is some $S_B \in \mathcal{M}_{i_1}$ with $S_A \subseteq S_B$. Since $\mathcal{M}_{i_1} = \mathcal{M}_{i_2}$, we know $S_B \in \mathcal{M}_{i_2} \subseteq \mathcal{C}_{i_2}$, so there is an integer $i_3 \in [n]$ with $i_2 < i_3$ and $S_B = S_C$, where $C = \{i_2, i_3\}$.

Now suppose $\psi(\{i_1, i_2, i_3\}) = \alpha$. Then $\alpha \in S_A \subseteq S_B = S_C$. So there exists $i_4 \in [n]$ with $i_4 > i_3$ such that $\psi(\{i_2, i_3, i_4\}) = \alpha$. This contradicts the assumption that ψ is a good coloring of $\mathbf{G}(n,3)$. We conclude that $\mathbf{2}^t$ has at least n antichains.

Now suppose that $\mathbf{2}^t$ contains at least n antichains. We show that $\chi(\mathbf{G}(n,3)) \leq t$. Let $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$ be antichains in $\mathbf{2}^t$ labelled so that if $1 \leq i_1 < i_2 \leq n$, then there exists a set $S \in \mathcal{M}_{i_2}$ so that $S \not\subseteq T$, for every $T \in \mathcal{M}_{i_1}$ (such a labelling is easily seen to exist).

For each $A = \{i_1, i_2\} \in \binom{[n]}{2}$, choose a set $S_A \in \mathcal{M}_{i_2}$ so that $S_A \not\subseteq T$ for every $T \in \mathcal{M}_{i_1}$. Then let $\{i_1, i_2, i_3\} \in \binom{[n]}{3}$, $A = \{i_1, i_2\}$ and $B = \{i_2, i_3\}$. Observe that $S_B \not\subseteq S_A$ so we may define a function $\psi : \binom{[n]}{3} \to [t]$ by taking $\psi(\{i_1, i_2, i_3\})$ as an element from $S_B - S_A$.

We claim that ψ is a good coloring of $\mathbf{G}(n,3)$. To see this, let $\{i_1, i_2, i_3, i_4\} \in \binom{[n]}{4}$, $A = \{i_1, i_2\}$, $B = \{i_2, i_3\}$, and $C = \{i_3, i_4\}$. Then $\psi(\{i_1, i_2, i_3\}) \in S_B - S_A$ and $\psi(\{i_2, i_3, i_4\}) \in S_C - S_B$, so $\psi(\{i_1, i_2, i_3\}) \neq \psi(\{i_2, i_3, i_4\})$. ■

The problem of counting the number of antichains in $\mathbf{2}^t$ is a classical one and is often called Dedekind's problem. Asymptotic solutions are given in [7] and [8]. For our purposes, we need only the fact that these estimates assert that the total number of antichains in $\mathbf{2}^t$ is approximately the number of subsets of the largest antichain.

Corollary 2.3. *The chromatic number of the shift graph* $\mathbf{G}(n,3)$ *satisfies:*

$$\chi(\mathbf{G}(n,3)) = \lg\lg n + \left(\frac{1}{2} + o(1)\right) \lg\lg\lg n. \quad ■$$

Further applications of shift graphs in the theory of chromatic numbers can be found in [3].

3. Interval orders and shift graphs

For an integer $n \geq 4$, let $\mathbf{I}_n = (I_n, P_n)$ denote the poset whose elements are the nondegenerate closed intervals with integer endpoints from $[n]$ with $[i_1, i_2] < [j_1, j_2]$ in P_n if and only if $i_2 < j_1$. Clearly, each $\mathbf{I}_n$ is an interval order, and we call the posets in the family $\{\mathbf{I}_n : n \geq 4\}$ *canonical* interval orders.

Theorem 3.1. *Let $n \geq 4$, let $\mathbf{I}_n = (I_n, P_n)$ be the canonical interval order and let $\mathbf{G}(n, 3)$ be the double shift graph. Then $\dim(\mathbf{I}_n) \geq \chi(\mathbf{G}(n, 3))$.*

Proof. Suppose that $\dim(\mathbf{I}_n) = t$. Choose linear extensions $L_1, L_2, \ldots, L_t$ of P_n whose intersection is P_n. Now define a coloring $\psi : \binom{[n]}{3} \to [t]$ as follows. For each $\{i_1, i_2, i_3\} \in \binom{[n]}{3}$, choose $\alpha \in [t]$ so that $[i_1, i_2] > [i_2, i_3]$ in L_α, and set $\psi(\{i_1, i_2, i_3\}) = \alpha$. If $\{i_1, i_2, i_3, i_4\} \in \binom{[n]}{4}$, there is no $\alpha \in [t]$ for which $\psi(\{i_1, i_2, i_3\}) = \alpha = \psi(\{i_2, i_3, i_4\})$, since this would require $[i_1, i_2] > [i_2, i_3] > [i_3, i_4]$ in L_α. However $[i_1, i_2] < [i_3, i_4]$ in P_n requires $[i_1, i_2] < [i_3, i_4]$ in L_α. We conclude that ψ is a good coloring of the shift graph $\mathbf{G}(n, 3)$, so $\dim(\mathbf{I}_n) \geq \chi(\mathbf{G}(n, 3))$. ∎

4. Some dimension theoretic preliminaries

For a poset $\mathbf{P} = (X, P)$, we write $x \| y$ in P and say x is *incomparable* to y in P when $x \not\leq y$ and $y \not\leq x$ in P. Then let $\operatorname{inc}(X, P) = \{(x, y) \in X \times X : x \| y \text{ in } P\}$. A subset $\{(x_i, y_i) : 1 \leq i \leq k\} \subseteq \operatorname{inc}(X, P)$ is called an *alternating cycle* (of length k) if $x_{i+1} \leq y_i$ for $i = 1, 2, \ldots, k-1$ and $x_1 \leq y_k$. An alternating cycle $\{(x_i, y_i) : 1 \leq i \leq k\}$ is *strict* if $x_j \leq y_i$ if and only if $j = i + 1$ for all $i = 1, 2, \ldots, k - 1$, and $x_j \leq y_k$ if and only if $j = 1$. In Figures 1a and 1b below, $\{(x_i, y_i) : 1 \leq i \leq 4\}$ is an alternating cycle of length 4. The cycle is strict in 1b but not in 1a.

The following elementary lemma is due to Trotter and Moore [16].

Lemma 4.1. *Let (X, P) be a poset and let $S \subseteq \operatorname{inc}(X, P)$. Then the following statements are equivalent:*
(1) *There is a linear extension L of P with $x > y$ in L for every $(x, y) \in S$.*
(2) *S does not contain any alternating cycles.*
(3) *S does not contain any strict alternating cycles.* ∎

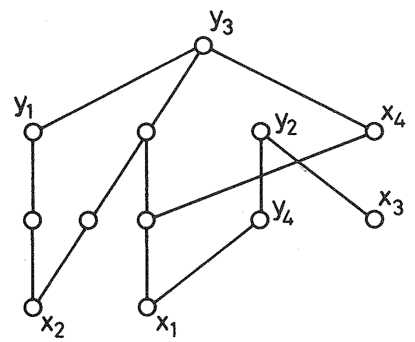

Figure 1a.

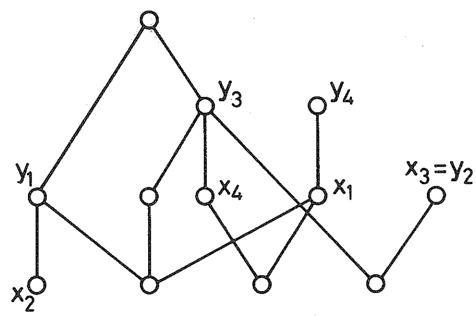

Figure 1b.

Let $\mathbf{P} = (X, P)$ be a poset. An incomparable pair (x, y) is called a *critical* pair if (1) $z < x$ in P implies $z < y$ in P for all $z \in X$ and (2) $y < w$ in P implies $x < w$ in P for all $w \in X$. It is easy to see that if $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ is a family of linear extensions of P, then $P = L_1 \cap L_2 \cap \ldots \cap L_t$ if and only if for every critical pair (x, y), there is some $\alpha \in [t]$ such that $x > y$ in L_α. Let $\text{crit}(X, P) = \text{crit}(\mathbf{P})$ denote the set of all critical pairs in the poset $\mathbf{P} = (X, P)$.

For a poset $\mathbf{P} = (X, P)$, define a hypergraph $\mathcal{H}_\mathbf{P}$ as follows. The vertex set of $\mathcal{H}_\mathbf{p}$ is $\text{crit}(\mathbf{P})$, the set of all critical pairs. The edge set of $\mathcal{H}_\mathbf{P}$ consists of those subsets of $\text{crit}(\mathbf{P})$ which form strict alternating cycles. For emphasis, we state the following result as a lemma, although it follows immediately from the definition of the hypergraph $\mathcal{H}_\mathbf{P}$.

Lemma 4.2. *Let $\mathcal{H}_\mathbf{P}$ be the hypergraph of critical pairs associated with a poset $\mathbf{P} = (X, P)$. Then $\chi(\mathcal{H}_\mathbf{P}) = \dim(\mathbf{P})$.* ■

The following lemma gives us a key property of edges in $\mathcal{H}_\mathbf{P}$ when $\mathbf{P}$ is an interval order.

Lemma 4.3. *Let $\mathbf{P} = (X, P)$ be an interval order and let $S = \{(x_i, y_i) : 1 \leq i \leq k\}$ be a strict alternating cycle of incomparable pairs. If $x_1 < y_k$, then $x_{i+1} = y_i$ for $i = 1, 2, \ldots, k-1$.*

Proof. Suppose $x_{i+1} < y_i$ for some i with $1 \le i \le k-1$. Then the four points x_1, y_k, x_{i+1}, y_i violate Fishburn's characterization (1.1) of interval orders. ■

Now let $\mathbf{P} = (X, P)$ be an interval order as evidenced by the map which assigns to each $x \in X$ a nondegenerate closed interval $[a_x, b_x]$ of $\mathbb{R}$ so that $x < y$ in P if and only if $b_x < a_y$ in $\mathbb{R}$. Set $\text{crit}^*(\mathbf{P}) = \{(x,y) \in \text{crit}(\mathbf{P}) : a_x < a_y \le b_x < b_y \text{ in } \mathbb{R}\}$ and $\dim^*(\mathbf{P})$ = least positive integer t for which there exist t linear extensions $L_1, L_2, \dots, L_t$ of P so that:

$(*)$ for every $(x,y) \in \text{crit}^*(\mathbf{P})$, there is some $j \in [t]$ such that $x > y$ in L_j.

Lemma 4.4. *If* $\mathbf{P} = (X, P)$ *is an interval order, then* $\dim^*(\mathbf{P}) \le \dim(\mathbf{P}) \le 2 + \dim^*(\mathbf{P})$.

Proof. The inequality $\dim^*(\mathbf{P}) \le \dim(\mathbf{P})$ is trivial. We now show $\dim(\mathbf{P}) \le 2 + \dim^*(\mathbf{P})$. Let $\dim^*(\mathbf{P}) = t$ and let $L_1, L_2, \dots, L_t$ be linear extensions of P satisfying property $(*)$. Then define linear extensions L_{t+1} and L_{t+2} by:

$$(1) \quad x < y \text{ in } L_{t+1} \text{ if } \begin{cases} a_x < a_y, \text{ or} \\ a_x = a_y \text{ and } x > y \text{ in } L_t \end{cases}$$

$$(2) \quad x < y \text{ in } L_{t+2} \text{ if } \begin{cases} b_x < b_y, \text{ or} \\ b_x = b_y \text{ and } x > y \text{ in } L_t \end{cases}$$

It follows easily that $P = L_1 \cap L_2 \cap \cdots \cap L_{t+2}$, so that $\dim(P) \le t+2$. ■

In view of the preceding lemma, we will be concerned only with estimating $\dim^*(\mathbf{P})$ in the remainder of this paper. For example, Theorem (3.1) actually asserts that $\dim^*(\mathbf{I}_n) \ge \chi(\mathbf{G}(n,3))$.

When $\mathbf{P} = (X, P)$ is an interval order, we let $\mathcal{H}^*_{\mathbf{P}}$ denote the subhypergraph of $\mathcal{H}_{\mathbf{P}}$ induced by $\text{crit}^*(\mathbf{P})$. Observe that $\dim^*(\mathbf{P}) = \chi(\mathcal{H}^*_{\mathbf{P}})$. For the canonical interval order $\mathbf{I}_n$, we abbreviate $\mathcal{H}_{\mathbf{I}_n}$ to $\mathcal{H}_n$ and $\mathcal{H}^*_{\mathbf{I}_n}$ to $\mathcal{H}^*_n$.

Lemma 4.5. *Let* $n \ge 4$. *Then a subset* $E \subseteq \text{crit}^*(\mathbf{I}_n)$ *is an edge in* $\mathcal{H}^*_n$ *if and only if there is some integer* $k \ge 2$ *and subsets* $\{a_1, a_2, \dots, a_{k+1}\}$, $\{b_1, b_2, \dots, b_{k+1}\} \in \binom{[n]}{k+1}$ *such that:*

1. $a_i < b_i$ *for* $i = 1, 2, \dots, k+1$,
2. $a_k \le b_1 < a_{k+1} \le b_2$ *and*
3. $E = \{([a_i, b_i], [a_{i+1}, b_{i+1}]) : 1 \le i \le k\}$.