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Colorful induced subgraphs

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Abstract

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A colored graph is a graph whose vertices have been properly, though not necessarily optimally colored, with integers. Colored graphs have a natural orientation in which edges are directed from the end point with smaller color to the end point with larger color. A subgraph of a colored graph is colorful if each of its vertices has a distinct color. We prove that there exists a function $f(k, n)$ such that for any colored graph G, if $\chi(G) > f(\omega(G), n)$ then G induces either a colorful out directed star with n leaves or a colorful directed path on n vertices. We also show that this result would be false if either alternative was omitted. Our results provide a solution to Problem 115, *Discrete Math.* 79.

1. Introduction

A triple $G = (V, E, f)$ is a *colored graph* (*digraph*) if (V, E) is a graph (digraph) and f is a proper vertex coloring of the graph (digraph) (V, E) with integers. The coloring f need not be optimal; in fact an important special case is that f is one-to-one. In this case we say that G is *colorful*. Let $G = (V, E, f)$ be a colored graph (digraph). The *natural orientation* of G is the colored digraph $NG = (V, A, f)$, with arc set $A = \{(x, y): xy \in E \text{ and } f(x) < f(y)\}$. Note that NG is an acyclic orientation of G. Let H be a subset of V. The colored subgraph (subdigraph) of G induced by H is $G[H] = (V, E', f')$, where (V, E') is the subgraph (subdigraph) of (V, E) induced by H and f' is f restricted to H. $G[H]$ is said to be an *induced colored subgraph of* G. We also say that G induces H' if H' is isomorphic to $G[H]$. We simplify notation by writing H for $G[H]$, when the meaning is clear from the context. Let DP_n denote the directed path on n vertices and DS_n denote the star $K_{1,n}$ oriented so that all edges are directed away from the vertex of degree n. Let $\omega(G)$ denote the clique number of (V, E) and $\chi(G)$ denote the chromatic number of G. Let $R(m, n)$ be the Ramsey function such that every graph on $R(m, n)$ vertices contains either a clique of size m or an independent set of size n. We prove the following theorems.

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Theorem 1. *There exists a function $h(k, n)$ such that for every colored graph $G=(V, E, f)$, if $\chi(G)>h(\omega(G), n)$ then the natural orientation NG induces either a colorful DS_n or a colorful DP_n.*

The following two theorems show that Theorem 1 cannot be strenghened by deleting either of the alternative conclusions.

Theorem 2. *For every natural number k, there exists a triangle free colored graph $G=(V, E, f)$ such that $\chi(G)=k$, but the natural orientation NG does not induce a colorful DS_2.*

We note that the graph G provided by Theorem 2 is not colorful. If G is colorful, then every induced subgraph of G is colorful. Thus, as Gyárfás pointed out, if G does not induce DS_n, then the out degree of G is bounded above by $b=R(\omega(G)+1, n)$, and thus $\chi(G)$ is bounded in terms of $\omega(G)$ and n by $2b+1$. Gyárfás [5] asked whether the chromatic number of an acyclicly oriented digraph G, which does not induce DP_4, is bounded in terms of $\omega(G)$. Since NG is acyclicly oriented, the next theorem answers this question negatively.

Theorem 3. *For every natural number k, there exists a triangle free, colored graph $G=(V, E, f)$ such that G is colorful and $\chi(G)=k$, but the natural orientation NG does not induce DP_4.*

It is worth noting other results on the chromatic number of graphs which do not induce various orientations of P_4. Chvátal [1] proved that an acyclicly oriented graph which does not induce $\longleftarrow\!\longrightarrow\!\longrightarrow$ (or $\longrightarrow\!\longrightarrow\!\longleftarrow$) is perfect. Gyárfás [5] points out that the shift graph $G(n, 2)$, introduced in the next section, which is triangle free and has chromatic number $\lceil \lg n \rceil$, can be acyclicly oriented so that it does not induce $\longleftarrow\!\longrightarrow\!\longleftarrow$. Kierstead [7] proved that the (on-line) chromatic number of an oriented graph which induces neither $\longleftarrow\!\longrightarrow\!\longrightarrow$, $\longrightarrow\!\longrightarrow\!\longleftarrow$, nor a directed 3-cycle, is bounded by $2^{\omega(G)}-1$.

Our interest in the questions addressed in this article arose from attempts to prove the following beautiful conjecture due independently to Gyárfás [3] and Sumner [10]. Let H be a graph and let forb(H) denote the class of graphs which do not induce H. The conjecture is that for every tree T, there exists a function f_t such that if $G \in$ forb(T), then $\chi(G)<f_T(\omega(G))$. Gyárfás, Szemerédi, and Tuza [6] have proved the special case of the conjecture where T has radius two and G is triangle free. Kierstead and Penrice [9] have recently removed the restriction that G be triangle free. Also see [4] and [8] for related results. We believe that our results may have applications to this conjecture.

2. Proofs

Let $G=(V, E, f)$ be a colored graph. For a vertex v, the colored out degree of v in G is $\operatorname{cod}_G(v)=|\{f(x)\colon vx\in E \text{ and } f(v)<f(x)\}|$. Let $\operatorname{cod}(G)=\max\{\operatorname{cod}_G(v)\colon v\in V\}$.

Proof of Theorem 1. Let $h=d^t$, where $d=R(\omega(G)+1, n)$ and $t=(d-1)(n-1)+1$. If $\operatorname{cod}(G)\geqslant d$, then NG induces a colorful DS_n, so assume $\operatorname{cod}(G)<d$. We define a coloring c on V such that $c(v)$ has the form $(c_1(v),\ldots, c_t(v))$, by recursion on i as follows. Let $c_0(v)=0$, for all $v\in V$. Suppose we have defined $c_j(v)$ for all $j\leqslant i$ and for all $v\in V$. Let $V(v, i)=\{w\in V\colon c_j(v)=c_j(w)$, for all $j\leqslant i\}$. Let $c_{i+1}(v)=\operatorname{cod}_{G[V(v,i)]}(v)$.

Clearly c is a d^t-coloring of G. The proof will be done if we show that either (1) c is a proper coloring of G, i.e., $V(v, t)$ is an independent set for all $v\in V$, or (2) G induces a colorful DP_n. If $c_t(v)=0$ then (1) clearly holds. Thus it suffices to show that if $c_t(v)>0$, then v is the first point of a colorful induced DP_n. Clearly $c_i(v)\geqslant c_{i+1}(v)$, for all i. Thus for some $i\leqslant t-n$, $c_{i+1}(v)=c_{i+2}(v)=\cdots=c_{i+n}(v)>0$. We shall actually show, by induction on s, that if $c_{i+1}(v)=c_{i+2}(v)=\cdots=c_{i+s}(v)>0$, then v is the first point of a colorful induced DP_s contained in $V(v, i)$.

Base Step: $s=1$. Trivial.

Inductive Step: $s=r+1$. Since $\operatorname{cod}(v)$ in $V(v, i+r)$ is at least one, there exists $w\in V(v, i+r)$ such that $vw\in E$ and $f(v)<f(w)$. Choose w so that $f(w)$ is as large as possible. Since $c_{i+1}(w)=c_{i+2}(w)=\cdots=c_{i+1}(w)=c_{i+r}(v)>0$, w is the first vertex of a colorful induced DP_r, say P, contained in $V(w, i)=V(v, i)$. Since $V(v, i+r)\subset V(v, i)$ and $\operatorname{cod}_{V(v,i)}(v)=\operatorname{cod}_{V(v,i+r)}(v)$, v is not adjacent to any vertex $x\in V(v, i)$ such that $f(x)>f(w)$. In particular, w is the only vertex of P which v is adjacent to. Thus $P+v$ is the desired colorful DP_s. □

For integers n and k, with $n>k$, Erdős and Hajnal [2] defined the *shift graph* $G(n, k)$ to be the graph whose vertices are the k-subsets of $\{1,\ldots, n\}$, where two vertices $X=\{x_1<\cdots<x_k\}$ and $Y=\{y_1<\cdots<y_k\}$ are adjacent iff $X\cap Y=\{x_2<\cdots<x_k\}=\{y_1<\cdots<y_{k-1}\}$ or vice versa. Clearly $\omega(G(n, k)=2$. Erdős and Hajnal proved that $\chi(G(n, k))=(1-o(1))\lg^{(k-1)} n$. In particular, $\chi(G(n, 2))=\lceil\lg n\rceil$, and if $\lg\lg n+\lg\lg\lg n>k$, then $\chi(G(n, 3))>k$.

Proof of Theorem 2. Fix a natural number k. Let $G=(V, E, f)$ be the colored graph such that $G(2^{2^{2k}}, 3)=(V, E)$ and $f(\{x_1<x_2<x_3\})=x_2$. Clearly f is a proper coloring of G. By the remarks above, $\omega(G)=2$ and $\chi(G)\geqslant k$. Consider a vertex $X=\{x_1<x_2<x_3\}$. If XY is an oriented edge in NG, then Y has the form $Y=\{x_2<x_3<y\}$, and thus $f(Y)=x_3$. We conclude that NG does not induce a colorful DS_2. □

To prove Theorem 3, we modify a construction of Zykov [11], which produces sparse triangle free graphs with large chromatic number. Our modification introduces new edges to eliminate induced DP_4's without increasing the clique size.

Proof of Theorem 3. We shall construct a sequence of colorful, colored graphs $G_i = (V_i, E_i, f_i)$ such that G_i is an induced subgraph of G_{i+1} and the vertices of $V_{i+1} - V_i$ receive lower colors than the vertices of V_i. In addition we will maintain a partition of the edges into red and blue edges so that:

(i) any two vertices are the end points of at most one red directed path;

(ii) all blue edges join vertices on red directed paths; and

(iii) the vertices on each red directed path induce a complete bipartite graph with red and blue edges.

We first show that (ii) and (iii) will ensure that $G = G_i$ triangle free and does not induce DP_4. First note that both an oriented triangle and DP_4 contain a directed Hamiltonian path. But if a subgraph H of G contains a directed Hamiltonian path, then by (ii), $V(H)$ is a subset of a red directed path, and by (iii), H is as complete bipartite subgraph of G. In paticular, H is not a triangle or DP_4.

Next we give the recursive construction of G. Let G_1 be the graph on one vertex. Now suppose we have constructed G_i. Let G_{i+1} consist of i independent copies (with distinct color sets) $G_i^j = (V_i^j, E_i^j)$ of G_i and a new $|V_i|^i$-set I_{i+1} of independent vertices, where $f(x) < f(v)$ for all vertices $x \in I_{i+1}$ and $v \in V_i^j$, $j = 1, \ldots, i$. For each i-tuple $(v^1, \ldots, v^i)$ with $v^j \in V_i^j$, choose $x \in I_{i+1}$ and join x to each v^j by a red edge. Then (i) will be satisfied. This creates some new directed red paths with initial vertex x. For each such path $P = (x = x_0, x_1, \ldots, x_r)$, join x to each x_{2k-1}, $2 \leq k \leq \lceil r/2 \rceil$, by a blue edge. This maintains (ii) and, by (i), does not violate (iii). The construction is now complete.

To see that $\chi(G_{i+1}) = i + 1$, note that any proper i-coloring of $G_{i+1} - I_{i+1}$ uses i distinct colors on each of V_i^j, for $j = 1, \ldots, i$, and thus some vertex of I_{i+1}, requires an additional color. This completes the proof. □

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