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Balance Theorems for Height-2 Posets

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Abstract. We prove that every height-2 finite poset with three or more points has an incomparable pair $\{x, y\}$ such that the proportion of all linear extensions of the poset in which x is less than y is between 1/3 and 2/3. A related result of Komlós says that the containment interval [1/3, 2/3] shrinks to [1/2, 1/2] in the limit as the width of height-2 posets becomes large. We conjecture that a poset denoted by V_m^+ maximizes the containment interval for height-2 posets of width $m + 1$.

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1. Introduction

Throughout this paper, a *poset* P is an ordered pair $(X, <_0)$ in which $<_0$ is an irreflexive and transitive binary relation on a finite set X of cardinality $n \geqslant 3$. We write $x \sim y$ if $x, y \in X, x \neq y$, and neither $x <_0 y$ nor $y <_0 x$. P is *linearly ordered* if $\sim$ is empty. A *linear extension of* $P = (X, <_0)$ is a linearly ordered set $(X, <_*)$ with $<_0 \subseteq <_*$, and when $x \sim y$ we define $p(x < y)$ by

$$p(x < y) = \frac{\text{number of linear extensions of } P \text{ in which } x <_* y}{\text{number of linear extensions of } P}.$$

P's *height* is the number of points in a maximum-cardinality linearly ordered subset of P, and its *width*, $w(P)$, is the number of points in a maximum-cardinality subset of X in which $<_0$ is empty.

For every poset P that is not linearly ordered, let

$$\delta(P) = \max_{x \sim y} \min\{p(x < y), p(y < x)\}$$

so that $0 < \delta(P) \leqslant 1/2$. We prove the first of the following theorems about $\delta(P)$ for height-2 posets.

THEOREM 1. *$\delta(P) \geqslant 1/3$ for every height-2 poset.*

THEOREM 2. $\lim_{m \to \infty} \min\{\delta(P)$: *P has height* 2, $w(P) = m\} = 1/2$.

The latter theorem was recently proved by Komlós [5] as part of a more general result for the limit 1/2.

Theorem 1 is motivated by the conjecture [3, 7] that $\delta(P) \geqslant 1/3$ for every nonlinear P: see Figure 1a. Kahn and Saks [4] prove $\delta(P) > 3/11$ for every nonlinear P. Linial [6] proves $\delta(P) \geqslant 1/3$ for every width-2 P, and Brightwell [2] does likewise for every nonlinear semiorder. Aigner [1] proves that the only width-2 P's with $\delta(P) = 1/3$ are ordinal sums (vertical stackings) of single points and the Figure 1a poset. He conjectures that $\delta(P) \neq 1/3$ for every P with $w(P) \geqslant 3$. Saks [7] reports that the smallest known $\delta(P)$ for width-3 posets is 14/39: see Figure 1b. A computer program of Gehrlein's for generating all small-n posets shows that no P with $w(P) \geqslant 3$ and $n \leqslant 9$ has a $\delta(P)$ smaller than 14/39.

Theorem 2 is motivated by the conjecture [4] for all posets that $\inf\{\delta(P): w(P) = m\} \to 1/2$ as $m \to \infty$. Komlós's proof of a specialization of this conjecture [5] is the first firm evidence for the general conjecture.

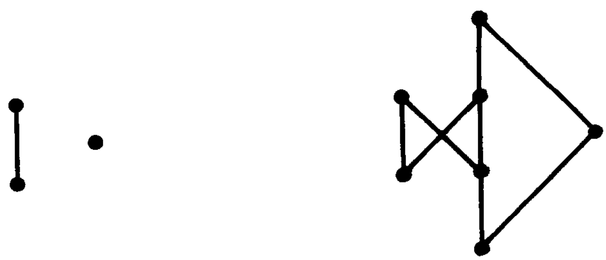
We have further results on the smallest $\delta(P)$ for height-2 posets for fixed n or w. Let V_m be the $2m$-point poset with m minimal points $l_1, l_2, \ldots, l_m$, m maximal points $u_1, u_2, \ldots, u_m$, and $l_i <_0 u_j \Leftrightarrow j \leqslant i$. Also let V_m^+ equal V_m plus an isolated point: see Figure 2. Let

$$\delta_n = \min\{\delta(P): P \text{ is an } n\text{-point height-2 poset}\}$$

$$\delta(m) = \min\{\delta(P): w(P) = m, P \text{ has height } 2\}.$$

We have verified

$$\delta(m+1) = \delta_{2m} = \delta_{2m+1} = \delta(V_m) = \delta(V_m^+)$$



$\delta(P) = 1/3$
(a)

$\delta(P) = 14/39$
(b)

Fig. 1.

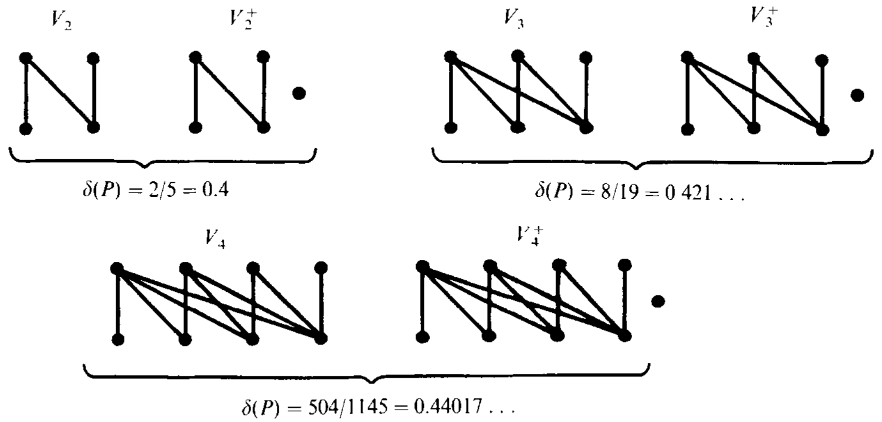


Fig. 2. δ maximizers.

for $m = 2, 3, 4$, and conjecture that it holds for all $m \geqslant 2$. Figure 2 shows the realizing posets. Further calculations give

$$\delta(V_5) = 15940/35505 = 0.4495\ldots$$

$$\delta(V_6) = 718050/1566813 = 0.4582\ldots$$

$$\delta(V_{10}) = 0.4748\ldots$$

$$\delta(V_{15}) = 0.4836\ldots.$$

The next section covers preliminaries that prepare for the proof of Theorem 1. The full proof appears in the final three sections.

2. Proof Preliminaries

For a height-2 poset $P = (X, <_0)$, let X_0 be the set of nonmaximal minimal points, and let X_1 be the set of nonminimal maximal points. Take $n_0 = |X_0|$ and $n_1 = |X_1|$. This leaves $n_2 = n - (n_0 + n_1)$ isolated points bearing $\sim$ to all others. If $n_2 \geqslant 1$ then $p(x < y)$ for $x, y \in X_0 \cup X_1$ is independent of the isolated points, so $\delta(P) \geqslant \delta(P$ with the isolates removed). For Theorem 1 it therefore suffices to prove that $\delta(P) > 1/3$ for every height-2 poset for which $n = n_0 + n_1 \geqslant 3$. In view of duality (inversion) we assume also that $n_1 \geqslant n_0$ and work henceforth with

$$\mathscr{P} = \{P : P \text{ has height } 2, \quad n = n_0 + n_1, \quad n_1 \geqslant n_0\}.$$

Let $\mathscr{L}$ denote the set of linear extensions of $P \in \mathscr{P}$. Taking the $L \in \mathscr{L}$ as equally likely, $p(E)$ for event E on $\mathscr{L}$ is the probability that E obtains. By prior notation, $p(x < y)$ is the probability that x is below y in $\mathscr{L}$.

Given $P \in \mathscr{P}$, for each $x \in X_1$ let $f(x)$ on $\mathscr{L}$ be the random quantity with value k at $L \in \mathscr{L}$ when exactly $k - 1$ points in X are below x in L. The probability that

x is in the top position is

$$t_x = p(f(x) = n).$$

Because every point in X_0 is covered by at least one point in X_1, $\Sigma_{X_1} t_x = 1$. Because every point in X_1 covers some point in X_0, $p(f(x) = 1) = 0$. Moreover,

$$t_x = p(f(x) = n) \geqslant p(f(x) = n - 1) \geqslant \cdots \geqslant p(f(x) = 2) \tag{1}$$

since $x \in X_1$ is maximal and can be interchanged with the point immediately above it in L to yield another $L' \in \mathscr{L}$ when it is not already on top. The preceding inequalities and the fact that $\Sigma_k p(f(x) = k) = 1$ imply

$$t_x \geqslant 1/(n-1) \quad \text{for every } x \in X_1.$$

For each $x \in X_1$ let

$$h(x) = \sum_{k=2}^{n} kp(f(x) = k), \quad [p(f(x) = 1) = 0]$$

the average "height" of x in $\mathscr{L}$. By (1), $h(x) \geqslant 2 + \Sigma_k (k-2)/(n-1) = 2 + (n-2)/2$. Also, by packing as much probability for f as possible near the top, we have $h(x) \leqslant t_x[n + (n-1) + \cdots + (n - q + 1)] + (1 - qt_x)(n - q)$, where $q = \lfloor 1/t_x \rfloor$. This gives

$$h(x) \leqslant n + \lfloor 1/t_x \rfloor(\lfloor 1/t_x \rfloor t_x/2 + t_x/2 - 1) \leqslant n + \tfrac{1}{2} - 1/(2t_x).$$

Therefore, for all $x \in X_1$,

$$n/2 + 1 \leqslant h(x) \leqslant n + \tfrac{1}{2} - 1/(2t_x). \tag{2}$$

These bounds provide information about $|h(y) - h(x)|$ that is used in the next two sections to verify $1/3 < p(x < y) < 2/3$ for all but the smallest (n_1, n_0) pairs.

3. Proof: Part 1

This section proves that if $P \in \mathscr{P}$ and (n_1, n_0) is *not* in

$$N = \{(8, 8), (7, 7), \ldots, (2, 2)\} \cup \{(7, 6), (6, 5), \ldots, (2, 1)\}$$

then $1/3 < p(x < y) < 2/3$ for some distinct $x, y \in X_1$. A tighter and more complex analysis in the next section shows that the same thing is true for the larger (n_1, n_0) in N. The remnant of smaller (n_1, n_0) in N is analyzed in the final section.

Given distinct $x, y \in X_1$, let

$$B = p(x < y) \quad \text{and} \quad b = p(f(y) - f(x) = 1) = p(f(x) - f(y) = 1),$$

where the p equality follows from interchanges of adjacent x and y in $\mathscr{L}$. We prove that

$$h(x) - h(y) > (1 - 2B - B^2)/(2b), \tag{3}$$

and will combine this with (2) shortly.

Let

$$a_k = p(f(x) - f(y) = k) \quad \text{and} \quad b_k = p(f(y) - f(x) = k)$$

for $k \geqslant 1$, so $B = p(x < y) = \Sigma\, b_k$, $1 - B = p(y < x) = \Sigma\, a_k$, $b = a_1 = b_1$ and

$$h(x) - h(y) = \sum_k k(a_k - b_k).$$

The final paragraph on p. 120 in Kahn and Saks [4] shows that, given fixed B and $b = a_1 = b_1$, $\Sigma\, k(a_k - b_k)$ is minimized by making the partial sums $a_1 + a_2$, $a_1 + a_2 + a_3, \ldots,$ as large as possible and by making the partial sums $b_1 + b_2, b_1 + b_3, \ldots,$ as small as possible.

Consider the a_k. Suppose $L \in \mathscr{L}$ has $f(x) - f(y) = k + 1$, $k \geqslant 1$. When y and the point immediately above it are interchanged, we get another linear extension for which $f(x) - f(y) = k$. This operation is one-one, so $b \geqslant a_2 \geqslant a_3 \geqslant \cdots$. Hence the partial a_k sums, beginning with a_1, can be no greater than $b, 2b, 3b, \ldots,$ until $1 - B$ is exhausted. Consider the b_k. As shown in Kahn and Saks [4], especially the proof of Lemma 2.6, the partial b_k sums can be no smaller than those of the geometric series $b, b(1 - b/B), b(1 - b/B)^2, \ldots,$ where $\Sigma_1^\infty b(1 - b/B)^{k-1} = B$.

Let $r = \lfloor (1 - B)/b \rfloor$. Then

$$\sum k a_k - \sum k b_k \geqslant \sum_{k=1}^{r} kb + (r+1)(1 - B - rb) - \sum_{k=1}^{\infty} kb(1 - b/B)^{k-1}.$$

Strict inequality holds if $n_1 \geqslant 3$ because of the infeasible tail in the later sum. That sum equals B^2/b. Let

$$S = \sum_{k=1}^{r} kb + (r+1)(1 - B - rb).$$

Observe that $r \leqslant (1 - B)/b \leqslant 2(1 - B)/b - 1$, hence that $1 - B - b/2 - br/2 \geqslant 0$, and that $-r \geqslant -(1 - B)/b$. Therefore

$$\begin{aligned} S &= r(1 - B - b/2 - br/2) + (1 - B) \\ &\geqslant [(1 - B)/b - 1](1 - B - b/2 - br/2) + (1 - B) \\ &\geqslant [(1 - B)/b - 1][1 - B - b/2 - (1 - B)/2] + (1 - B) \\ &= (1 - B)^2/(2b) + b/2 \\ &> (1 - B)^2/(2b). \end{aligned}$$

Thus $\Sigma\, k(a_k - b_k) > (1 - B)^2/(2b) - B^2/b = (1 - 2B - B^2)/(2b)$, and this verifies (3).

Since $1 - 2B - B^2$ decreases in B and equals 0 at $B = \sqrt{2} - 1$, it follows immediately from (3) that if $0 \geqslant h(x) - h(y)$ then $B > \sqrt{2} - 1$. Equivalently, for all $x, y \in X_1$,

$$h(x) \geqslant h(y) \Rightarrow p(x < y) < 2 - \sqrt{2} < 2/3. \tag{4}$$

Moreover, since (3) says that $b[h(x) - h(y)] > (1 - 2B - B^2)/2$, and since $(1 - 2B - B^2)/2 = 1/9$ when $B = 1/3$,

$$h(x) \geqslant h(y) \quad \text{and} \quad b[h(x) - h(y)] \leqslant 1/9 \Rightarrow p(x < y) > 1/3. \tag{5}$$

We use (2) to show that the hypotheses of (5) hold for some $x, y \in X_1$ when $(n_1, n_0) \notin N$.

For convenience henceforth let $m = n_1$, so $m \geqslant n/2$. Also let $X_1 = \{1, 2, \dots, m\}$ and without loss of generality suppose that

$$1/(n-1) \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_m.$$

Fix k in $\{2, \dots, m\}$. By (2), $h(1)$ through $h(k)$ all lie in $[n/2 + 1, n + \frac{1}{2} - 1/(2t_k)]$. Therefore, regardless of the ordering of $h(1), \dots, h(k)$ within this interval, there are distinct $i, j \leqslant k$ such that

$$h(i) \geqslant h(j) \quad \text{and} \quad h(i) - h(j) \leqslant \frac{(n-1)/2 - 1/(2t_k)}{k-1}.$$

Since $p(f(i) - f(j) = 1) \leqslant t_i$, as seen by moving maximal i into the top position of L whenever $f(i) - f(j) = 1$, we have

$$b \leqslant \min\{t_i, t_j\}. \tag{6}$$

In particular, $b \leqslant t_k$, so $b[h(i) - h(j)] \leqslant [(n-1)t_k - 1]/[2(k-1)]$.

It follows that there are $x, y \in X_1$ such that

$$h(x) \geqslant h(y) \quad \text{and} \quad b[h(x) - h(y)] \leqslant \min_{2 \leqslant k \leqslant m} \frac{(n-1)t_k - 1}{2(k-1)}.$$

Let $Z = \min\{[(n-1)t_k - 1]/[2(k-1)]\}$. When t_1 is fixed, it is easily seen that Z is maximized when the min arguments are equal, or when $Z2(k-1) = (n-1)t_k - 1$ for $k = 2, \dots, m$. Summation yields $Z(m-1)m = (n-1)(1-t_1) - (m-1)$, so Z is maximized globally at min $t_1 = 1/(n-1)$. Hence $Z \leqslant (n-2)/[m(m-1)] - 1/m$.

Therefore there are distinct $x, y \in X_1$ such that

$$h(x) \geqslant h(y) \quad \text{and} \quad b[h(x) - h(y)] \leqslant \frac{n-m-1}{m(m-1)}. \tag{7}$$

Given (7), $p(x < y) > 2/3$ by (4). By (5), $1/3 < p(x < y)$ if

$$\frac{n-m-1}{m(m-1)} \leqslant \frac{1}{9}.$$

Given $n = m + n_0$ and $m = n_1 \geqslant n_0$, it is routinely checked that this inequality holds except for $(n_1, n_0) \in N$.

4. Proof: Part 2

We modify the preceding proof after (6) to obtain the desired result for the larger (n_1, n_0) pairs in N.

Fix $k \in \{2, \ldots, m\}$ as in the paragraph of (6). Suppose

$$h(\sigma_1) \leqslant h(\sigma_2) \leqslant \cdots \leqslant h(\sigma_k),$$

where $\sigma_1, \sigma_2, \ldots, \sigma_k$ is a rearrangement of $1, 2, \ldots, k$. Then, with b as in (6),

$$b[h(\sigma_{i+1}) - h(\sigma_i)] \leqslant \min\{t_{\sigma_i}, t_{\sigma_{i+1}}\}[h(\sigma_{i+1}) - h(\sigma_i)],$$

for $i = 1, \ldots, k-1$. Let $s_i = \min\{t_{\sigma_i}, t_{\sigma_{i+1}}\}$ and $d_i = h(\sigma_{i+1}) - h(\sigma_i) \geqslant 0$ for $1 \leqslant i \leqslant k-1$. Then

$$\min_{1 \leqslant i < k} b[h(\sigma_{i+1}) - h(\sigma_i)] \leqslant \min\{s_1 d_1, \ldots, s_{k-1} d_{k-1}\}$$

with $\Sigma\, d_i \leqslant (n-1)/2 - 1/(2t_k)$ by (2). Sequence $s_1, s_2, \ldots, s_{k-1}$ has t_1 at least once, t_1 or t_2 at least twice, ..., so

$$\min_{1 \leqslant i < k} b[h(\sigma_{i+1}) - h(\sigma_i)] \leqslant \max_{(d)} \min\{t_1 d_1, \ldots, t_{k-1} d_{k-1}\}, \tag{8}$$

where (d) denotes the set of all nonnegative sequences $d_1, \ldots, d_{k-1}$ whose terms sum to $(n-1)/2 - 1/(2t_k)$. Therefore $\max_{(d)} \min\{t_i d_i\}$ is realized when $t_1 d_1 = t_2 d_2 = \cdots = t_{k-1} d_{k-1}$. If $k = 2$, $\max_{(d)} \min\{t_i d_i\} = [(n-1)/2 - 1/(2t_2)]t_1 = [n-1)/2 - 1/(2t_2)]/(1/t_1)$; if $k \geqslant 3$,

$$\max_{(d)} \min\{t_i d_i\} = [(n-1)/2 - 1/(2t_k)]/(1/t_1 + 1/t_2 + \cdots + 1/t_{k-1}).$$

Let $v_2, v_3, \ldots, v_m$ be twice the $\max_{(d)}$ min values at $k = 2, 3, \ldots, m$ respectively, and let

$$q_i = 1/t_i \quad \text{for} \quad i = 1, 2, \ldots, m.$$

Then, by the preceding paragraph,

$$\begin{aligned} q_1 v_2 + q_2 &= n-1 \\ (q_1 + q_2)v_3 + q_3 &= n-1 \\ (q_1 + q_2 + q_3)v_4 + q_4 &= n-1 \\ &\vdots \\ (q_1 + \cdots + q_{m-1})v_m + q_m &= n-1. \end{aligned} \tag{9}$$

Moreover, by (8), there are $x, y \in X_1$ such that

$$h(x) \geqslant h(y) \quad \text{and} \quad b[h(x) - h(y)] \leqslant \min\{v_2, v_3, \ldots, v_m\}/2.$$

If $\min\{v_2, \ldots, v_m\}/2 \leqslant 1/9$ also, then $1/3 < p(x < y) < 2/3$ as in the analysis following (7).

Let

$$V = \max_{(q)} \min\{v_2, \ldots, v_m\},$$

where (q) denotes the set of all sequences $q_1, q_2, \ldots, q_m$ that satisfy (9) subject to

$$(n-1) \geqslant q_1 \geqslant q_2 \geqslant \cdots \geqslant q_m > 0 \quad \text{and} \quad \sum_{i=1}^{m} 1/q_i = 1. \tag{10}$$

Because of the nonlinearity caused by $\Sigma\, 1/q_i = 1$, determination of V is more complex than the determination of $\max Z$ that precedes (7).

An analysis of (9) subject to (10) shows that V obtains when one of the following three things holds:

[A] $v_2 > v_3 = v_4 = \cdots = v_m$ with $q_1 = q_2$;
[B] $v_2 = v_3 = \cdots = v_m$;
[C] $q_1 = q_2 = \cdots = q_m$.

In particular, if neither [A] nor [B] holds, then either

[D] $v_k > v_j$ for some $j, k \geqslant 3$, or
[E] $v_2 > v_3 = \cdots = v_m$ and $q_1 > q_2$, or
[F] $v_2 < v_3 = \cdots = v_m$,

and in each a change in q that satisfies (9) and (10) will increase $\min\{v_2, \ldots, v_m\}$, except perhaps when [C] obtains. For example, min increases under [E] when q_1 and q_2 are moved closer together: with $1/q_1 + 1/q_2 = c$, $q_1 + q_2 = q_1 + q_1/(cq_1 - 1)$; the derivative of the latter expression with respect to q_1 is positive since $cq_1 > 2$ given $q_1 > q_2$; hence $q_1 + q_2$ decreases when q_1 decreases and, by (9), this forces each of v_3 through v_m to increase. Similarly, if [F] holds, v_2 will increase as we move q_1 and q_2 farther apart: we cannot have $q_2 = q_3$ to begin with since this implies that $v_2 > v_3$. But we might have $q_1 = n - 1$ for [F], in which case a decrease in q_1, or in both q_1 and q_2 if $q_1 = q_2$, and a compensating increase in q_m will increase v_2.

Suppose [D] obtains. If $q_1 > q_2$ and $v_k > \min$ for $k \geqslant 3$, we increase every v_i other than v_k by decreasing q_1 and increasing q_k. This move is feasible unless $q_{k-1} = q_k$, in which case $v_{k-1} > v_k$, and continuation with $k - 1$ in place of k leads to the conclusion that we increase min unless $q_1 > q_2 = \cdots = q_k$, which requires $v_2 > v_3 > \cdots > v_k$. But then the move described for [E] increases min. On the other hand, if [D] obtains and $q_1 = q_2$, we can increase $\min\{v_2, \ldots, v_m\}$ unless perhaps

$$q_1 = q_2 = \cdots = q_{m-1} \geqslant q_m.$$

Further analysis shows that we can do no better here than to take $q_{m-1} = q_m$, which gives [C]: we omit the details.

Suppose henceforth in this section that one of [A], [B] and [C] holds along with (9) and (10). As noted after (9), if $V \leqslant 2/9$ then $1/3 < p(x < y) < 2/3$ for some $x, y \in X_1$. It turns out for the (m, n_0) cases in N, that [A] yields V. The $\max\min\{v_2, \ldots, v_m\}$ values under [B] are smaller than those under [A], and the values under [C] are smaller than those under [B]. We describe the analysis for [A] and [C]. The analysis for [B] is similar to that for [A].

Suppose [C] holds. Then $q_i = m$ for all i, and

$$\min\{v_2, \ldots, v_m\} = v_m = (n-1-m)/[m(m-1)].$$

It follows that $v_m \leqslant 2/9$ if and only if $9(n-1) \leqslant 2m^2 + 7m$. Since $n \leqslant 2m$, i.e., $n_1 \geqslant n_0$, $v_m \leqslant 2/9$ whenever $m \geqslant 2$. Therefore, given [C], all pairs in N have $v_m \leqslant 2/9$.

Suppose [A] holds. Let

$$q = q_1 = q_2, \quad v = v_3 = \cdots = v_m = \min\{v_i\}, \quad \beta = 1/(1-v).$$

The equations of (9) yield

$$n - 1 = q(1 + v_2), \quad q_3 = n - 1 - 2qv, \quad q_k = q_3(1-v)^{k-3} \quad \text{for} \quad k \geqslant 4.$$

By $\Sigma\,(1/q_i) = 1$ and $v = (\beta - 1)/\beta$, we have

$$1 = \frac{2}{q} + \frac{\beta^{m-2} - 1}{(n-1-2vq)(\beta - 1)} = \frac{2}{q} + \frac{\gamma}{(n-1)\beta - 2(\beta - 1)q},$$

where $\gamma = \beta(\beta^{m-2} - 1)/(\beta - 1)$. This gives a quadratic equation in q whose solutions are

$$q = \frac{(n-1)\beta + 4(\beta-1) - \gamma \pm [((n-1)\beta + 4(\beta - 1) - \gamma)^2 - 16(n-1)\beta(\beta-1)]^{1/2}}{4(\beta - 1)}.$$

Since $q \geqslant q_3 = n - 1 - 2qv$, we require $q \geqslant (n-1)/(1+2v)$. Analysis then shows that, when $m \geqslant 6$ and v is in the neighborhood of 2/9, we must use the + root of the quadratic solution. With that root, $q \geqslant (n-1)/(1+2v)$ reduces to

$$n - 1 \geqslant \frac{(3\beta - 2)(\beta^{m-2} + 2\beta - 3)}{\beta(\beta - 1)}. \tag{11}$$

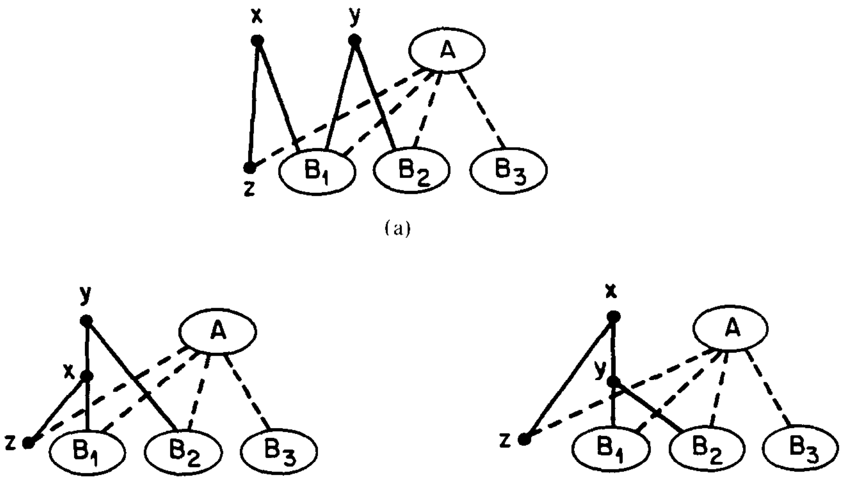
When $m \geqslant 6$ and v is in the neighborhood of 2/9, or larger, the right side of the preceding inequality increases in β, or in v since v increases as β increases. Thus, to avoid the conclusion that $v \leqslant 2/9$, hence that $1/3 < p(x < y) < 2/3$ for some $x, y \in X_1$, the preceding inequality must hold when $\beta = 9/7$, i.e., when $v = 2/9$. To avoid the desired conclusion, calculations at $\beta = 9/7$ show that if $m = 6$ then $n - 1 \geqslant 12$ and, in general, if $m \geqslant 6$ then $n > 2m$. Since we require $n \leqslant 2m$, the desired result always hold if $m \geqslant 6$. Similar results hold for case [B].

When these conclusions are combined with those in the preceding section, we see that $1/3 < p(x < y) < 2/3$ for some $x, y \in X_1$ except perhaps when (n_1, n_0) is in

$$N^* = \{(5,5), (5,4), (5,3), (4,4), \ldots, (2,1)\}.$$

5. Proof: Part 3

The results for V_m and V_m^+ in the penultimate paragraph of the introduction cover all pairs in N^* except (5, 5). We conclude the proof of Theorem 1 by applying the following lemma to (5, 5).



(a)

(b) $x < y$ (c) $y < x$

Fig. 3. $p(x < y) > 1/3$.

LEMMA 1. *Suppose* $P \in \mathscr{P}$. *If* $x, y \in X_1$ *and* $p(x < y) \leqslant 1/3$ *then x must cover at least two points in* X_0 *not covered by y. If* $x, y \in X_0$ *and* $p(x < y) \geqslant 1/3$, *then y must be covered by at least two points in* X_1 *that do not cover x.*

Proof. We prove only the first part since the other proof is similar. Suppose $x, y \in X_1$. It is easily seen that $p(x < y) \geqslant 1/2$ if x covers no point not covered by y. Suppose that x covers exactly one point $z \in X_0$ that is not covered by y: see Figure 3a, where $X_1 = \{x, y\} \cup A$, $X_0 = \{z\} \cup B_1 \cup B_2 \cup B_3$ and $B_1 \cup B_2 \neq \emptyset$. Dashed lines indicate possible covers.

The modified diagrams for $x < y$ and $y < x$ are shown in the lower part of the figure. Let b_1, c_1 and c_2 denote the number of linear extensions of (b), of (c) when $z < y$, and of (c) when $y < z < x$, respectively. We claim that $c_2 < c_1 \leqslant b_1$, from which it follows that

$$p(x < y) = \frac{b_1}{b_1 + c_1 + c_2} > \frac{1}{3}.$$

Consequently, $p(x < y) \leqslant 1/3$ forces x to cover at least two points in X_0 not covered by y. We now prove the claim.

Suppose $z < y$ in (c). If $B_2 = \emptyset$ then $c_1 = b_1$ since (b) and (c) with $z < y$ are identical up to the x, y labels. If $B_2 \neq \emptyset$ then $c_1 < b_1$ since $c_1 = b_1'$ when b_1' is the number of extensions for (b) that have $B_2 < x$. Hence $c_1 \leqslant b_1$.

Suppose (c) obtains. If $B_1 = \emptyset$ then $c_2 < c_1$ since a proper subset of the set of linear extensions for c_1 is isomorphic by restrictions of the diagram for c_1 to the set of linear extensions for c_2. If $B_1 \neq \emptyset$, let c_3 be the number of linear extensions with $B_1 < z < y$. Then $c_3 < c_1$ since there are extensions with z below points in B_1, and $c_2 \leqslant c_3$ by subset isomorphism. Threrfore $c_2 < c_1$. ■

We now analyze $(n_1, n_0) = (5, 5)$ *for* $P \in \mathscr{P}$. We suppose that $1/3 < p(x < y) < 2/3$ never occurs and proceed to a contradiction.

Given this suppostion for (5, 5) let $X_1 = \{1, 2, \ldots, 5\}$ and $X_0 = \{x_1, x_2, \ldots, x_5\}$. For $i, j \in X_1$, let $i >_2 j$ mean that i covers at least two points in X_0 not covered by j. For $x, y \in X_0$, let $x <_2 y$ mean that y is covered by at least two points in X_1 that do not cover x. By our supposition, if $a, b, c \in X_1$ or $a, b, c \in X_0$ then $[p(a < b) \leqslant 1/3, p(b < c) \leqslant 1/3] \Rightarrow p(a < c) > 2/3 \Rightarrow p(a < c) \leqslant 1/3$. Therefore, by Lemma 1, we assume without loss of generality that $i >_2 j$ whenever $1 \leqslant i < j \leqslant 5$. Similarly, there is a linear arrangement of the points in X_0 such that $x <_2 y$ whenever x precedes y in the arrangement.

Since $5 \in X_1$ covers something in X_0, suppose for definiteness that $x_1 <_0 5$. By $4 >_2 5$, 4 covers two points in X_0 that differ from x_1: call them x_2 and x_3. Assume $x_2 <_2 x_3$ for definiteness. Then x_3 is covered by two points in X_1, say a and b, that don't cover x_2. Since 5 doesn't cover x_3, and 4 covers both x_2 and x_3, $\{a, b\} \cap \{4, 5\} - \emptyset$. Assume $a >_2 b$ for definiteness. One of the points in X_0 covered by a and not b must differ from x_1, x_2 and x_3: call it x_4. The other X_0 point for $a >_2 b$ can also be new (x_5) or it can be x_1. However, because b covers two points in X_0 not covered by 4, this forces a sixth point in X_0. Since this contradicts $|X_0| = 5$, the proof for (5, 5) is complete.

6. Discussion

We have shown that every height-2 poset with $n \geqslant 4$ has an incomparable pair for which $1/3 < p(x < y) < 2/3$. The smallest known $\delta(P)$ for such posets is 2/5, which obtains for V_2 and V_2^+. It is almost certainly true that every height-2 P with $n \geqslant 6$ has an incomparable pair for which $2/5 < p(x < y) < 3/5$, but our approach only verifies this for large n and for very small n. In comparison with Theorem 2, which implies that $\delta(P)$ is arbitrarily close to 1/2 when n is suitably large, our methods show only that $\delta(P)$ is at least as large as $\sqrt{2} - 1 - \varepsilon$ when n is large.

Two fundamental open questions about δ for height-2 posets concern its minimum value $\delta(m)$ for width-m posets, and the actual forms of the posets that attain this minimum.

Q1. Is $\delta(m)$ nondecreasing in m?

Q2. Does $\delta(m + 1) = \delta(V_m^+)$ and, if so, is V_m^+ the unique realizer of $\delta(m + 1)$?

A positive answer to the first part of Q2 would answer Q1 in the affirmative.

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