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Balancing Pairs and the Cross Product Conjecture*

G. R. BRIGHTWELL
Department of Mathematics, London School of Economics, Houghton Street, London WC2A 2AE, U.K.
E-mail address: grb10@phoenix.cambridge.ac.uk

S. FELSNER
Freie Universität Berlin, Fachbereich Mathematik, Institut für Informatik, Takustr. 9, 14195 Berlin, Germany
E-mail address: felsner@inf.fu-berlin.de

and

W. T. TROTTER
Department of Mathematics, Arizona State University, Tempe AZ 85287, U.S.A.
E-mail address: trotter@erdos.la.asu.edu

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Abstract. In a finite partially ordered set, $\text{Prob}\,(x > y)$ denotes the proportion of linear extensions in which element x appears above element y. In 1969, S. S. Kislitsyn conjectured that in every finite poset which is not a chain, there exists a pair (x, y) for which $1/3 \leqslant \text{Prob}\,(x > y) \leqslant 2/3$. In 1984, J. Kahn and M. Saks showed that there exists a pair (x, y) with $3/11 < \text{Prob}\,(x > y) < 8/11$, but the full 1/3–2/3 conjecture remains open and has been listed among ORDER's featured unsolved problems for more than 10 years.

In this paper, we show that there exists a pair (x, y) for which $(5-\sqrt{5})/10 \leqslant \text{Prob}\,(x > y) \leqslant (5+\sqrt{5})/10$. The proof depends on an application of the Ahlswede–Daykin inequality to prove a special case of a conjecture which we call the Cross Product Conjecture. Our proof also requires the full force of the Kahn–Saks approach – in particular, it requires the Alexandrov–Fenchel inequalities for mixed volumes.

We extend our result on balancing pairs to a class of countably infinite partially ordered sets where the 1/3–2/3 conjecture is *false*, and our bound is best possible. Finally, we obtain improved bounds for the time required to sort using comparisons in the presence of partial information.

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Key words. Partially ordered set, linear extension, balancing pairs, cross-product conjecture, Ahlswede–Daykin inequality, sorting.

* An extended abstract of an earlier version of this paper appears as [6]. The results here are much stronger than in [6], and this paper has been written so as to overlap as little as possible with that version.

1. Introduction

Given a finite partially ordered set (poset) P, let $\Lambda(P)$ denote the set of linear extensions of P, and let $L(P) = |\Lambda(P)|$. For a pair x, y of distinct elements of P, let $\mathrm{Prob}\,(x > y)$ denote the number of linear extensions of P in which $x > y$, divided by $L(P)$. Thus $\mathrm{Prob}\,(x > y)$ is the proportion of linear extensions in which x is above y. The probabilistic notation is of course quite natural, corresponding to making $\Lambda(P)$ into a probability space with each linear extension equally likely. If $x < y$ in P, then $\mathrm{Prob}\,(x > y) = 0$, while $\mathrm{Prob}\,(x > y) = 1$ if $x > y$ in P. On the other hand, if x and y are incomparable in P, then $0 < \mathrm{Prob}\,(x > y) < 1$. In 1969, S. S. Kislitsyn [15] made the following conjecture, which remains one of the most intriguing problems in the combinatorial theory of posets.

CONJECTURE 1.1. *If P is a finite poset which is not a chain, then there exists an incomparable pair $x, y \in P$ so that*

$$1/3 \leqslant \mathrm{Prob}\,(x > y) \leqslant 2/3.$$

Conjecture 1.1 was also made independently by both M. Fredman and N. Linial, and many papers on this subject attribute the conjecture to them. It is now known as the 1/3–2/3 conjecture. If true, the conjecture would be best possible, as shown by the poset with three elements and one comparable pair.

The 1/3–2/3 conjecture has been proved for several special classes of posets. Linial [17] showed that the conjecture holds for width two posets, and P. Fishburn, W. G. Gehrlein and W. T. Trotter [8] showed that it holds for height two posets. G. Brightwell [3] showed that it holds for semiorders, and Brightwell and C. D. Wright [5] verified it for posets in which every element is incomparable with at most five others.

Following Kahn and Saks, for a finite partially ordered set P, we let $\delta(P)$ denote the largest positive real number so that there exists a pair (x, y) of distinct points from P with $\delta(P) \leqslant \mathrm{Prob}\,(x > y) \leqslant 1 - \delta(P)$. We may then set δ_0 to be the infinum of $\delta(P)$, taken over all finite P which are not chains. With this notation, the 1/3–2/3 conjecture is just the assertion that $\delta_0 \geqslant 1/3$. However, to the best of our knowledge, there is no entirely elementary proof that $\delta_0 > 0$.

The first major breakthrough in this area came in 1984, when Kahn and Saks [13] used the Alexandrov–Fenchel inequalities for mixed volumes to prove the following result.

THEOREM 1.2. *If P is a finite poset which is not a chain, then there exists an incomparable pair $x, y \in P$ so that*

$$3/11 < \mathrm{Prob}\,(x > y) < 8/11.$$

Thus, $\delta_0 > 3/11 \simeq 0.2727$. □

Our main result is the following inequality which improves the bound in Theorem 1.2.

THEOREM 1.3. *If P is a finite poset which is not a chain, then there exists an incomparable pair $x, y \in P$ so that*

$$(5-\sqrt{5})/10 \leqslant \text{Prob}\,(x > y) \leqslant (5+\sqrt{5})/10.$$

Thus, $\delta_0 \geqslant (5-\sqrt{5})/10 \simeq 0.2764$.

The proof of Theorem 1.3 requires all the machinery developed by Kahn and Saks for the proof of Theorem 1.2. In particular, it requires the Alexandrov/Fenchel inequalities for mixed volumes to prove that certain sequences are log-concave.

Our proof also requires a new inequality – a special case of a conjecture which we call the *Cross Product Conjecture*. Although we have not been able to settle the Cross Product Conjecture in full generality, the special case is enough for our purposes here. Even this case requires an application of the Ahlswede/Daykin inequality – a deep and powerful combinatorial tool which has already found a wide range of applications to posets.

Numerically, Theorem 1.3 is only a modest improvement on the Kahn/Saks bound and leaves us far short of settling the $1/3$–$2/3$ conjecture. But in a certain sense, our Theorem 1.3 is best possible. To explain this remark, we extend our investigation to include countably infinite posets, and for this class of posets, the $1/3$–$2/3$ conjecture is false. Call a poset P *thin* if there is some natural number k such that every element of P is incomparable with at most k others. If a thin poset has a connected incomparability graph, then it is countable, and each interval $[a, b] \equiv \{z\colon\ a \leqslant z \leqslant b\}$ is finite. Let $([a_n, b_n])$ be a nested sequence of intervals whose union is the ground-set of P. If the elements x, y of P lie in one of the intervals $[a_m, b_m]$, then for $n \geqslant m$ we can consider $\text{Prob}_n(x > y) \equiv \text{Prob}(x > y)$ in the poset $P|[a_n, b_n]$. Brightwell [2] showed that $\lim_{n\to\infty} \text{Prob}_n(x > y)$ exists, and is independent of the sequence of intervals chosen. We naturally define $\text{Prob}\,(x > y)$ in P to be this limit. This definition can be extended in an obvious way to thin posets with disconnected incomparability graph.

For a thin poset P, we again define $\delta(P)$ to be the largest positive number for which there exists a pair $x, y \in P$ with $\delta(P) \leqslant \text{Prob}\,(x > y) \leqslant 1 - \delta(P)$. Now let δ_0' be the infimum of $\delta(P)$ over all thin posets P other than chains.

As was discovered independently by Brightwell [2] and Trotter, there is a thin poset Q with $\delta(Q) = (5-\sqrt{5})/10$. This example is constructed as follows. The poset Q has as its point set $X = \{x_i\colon\ i \in \mathbb{Z}\}$ with: $x_i < x_j$ in Q if and only if $j > i + 1$ in $\mathbb{Z}$. If we define the finite poset Q_n to be the subset of Q consisting of all points whose subscripts in absolute value are at most n, then it is an easy exercise to show that

$$\lim_{n\to\infty} \text{Prob}\,(x_0 > x_1) = (5-\sqrt{5})/10.$$

Thus $\delta_0' \leqslant (5-\sqrt{5})/10$. On the other hand, our proof of Theorem 1.3 works in the infinite setting, so we obtain the following result, proving a conjecture of Brightwell [2] and Trotter.

THEOREM 1.4. $\delta_0' = (5-\sqrt{5})/10 \simeq 0.2764$.

The example Q is striking on several counts. Observe that Q has width two, is a semiorder, and each of its elements is incomparable with just two others. As noted above, any *finite* poset P satisfying any one of these three conditions would have $\delta(P) \geqslant 1/3$.

Prior to 1994, the Kahn–Saks bound given in Theorem 1.2 was the best known bound known lower bound on δ_0 valid for all finite posets. However, other proofs bounding δ_0 away from zero have been given. In [14], L. Khachiyan uses geometric techniques to show $\delta_0 \geqslant 1/e^2$. Kahn and Linial [12] provide a short and elegant argument using the Brunn–Minkowski theorem to show that $\delta_0 \geqslant 1/2e$. In [10], J. Friedman also applies geometric techniques to obtain even better constants when the poset satisfies certain additional properties. In [6], Felsner and Trotter showed that there exists an absolute constant $\varepsilon > 0$ so that $\delta_0 \geqslant 3/11 + \varepsilon$.

Kahn and Saks conjectured that $\delta(P)$ approaches $1/2$ as the width of P tends to infinity. In [16], J. Komlós provides support for this conjecture by showing that for every $\varepsilon > 0$, there exists a function $f_\varepsilon(n) = o(n)$ so that if $|P| = n$ and P has at least $f_\varepsilon(n)$ minimal points, then $\delta(P) > 1/2 - \varepsilon$.

The remainder of the paper is organized as follows. In Section 2, we outline the basic flow of the proof of Theorem 1.3. In Section 3, we present the Cross-Product Conjecture, and the proof of the special case of the conjecture necessary for this paper. The main body of the proof is given in Sections 4, 5 and 6. In Section 7, we provide additional details on the class of countably infinite posets where our Theorem 1.3 holds and is best possible. Finally, in Section 8, we produce a new bound for sorting using comparisons – the motivating problem for the study of balancing pairs.

2. The Basic Approach

For the next three sections of this paper, we shall deal exclusively with finite posets. Our aim is to develop the machinery necessary to prove Theorem 1.3. Fundamentally, our method is a both a refinement and an extension of that used by Kahn and Saks in [13]. Accordingly, we will use the notation and terminology of that paper as far as possible.

Throughout, we consider the sample space of all linear extensions of a finite poset P, with all linear extensions being equally likely. For a linear extension λ and a point $x \in P$, $h_\lambda(x)$ denotes the *height* of x in λ, i.e., if λ orders the points in P as $x_1 < x_2 < \cdots < x_n$ and $x = x_i$, then $h_\lambda(x) = i$. We denote by $h(x)$ the expected value of $h_\lambda(x)$. When $(x, y) \in P \times P$ is a fixed ordered pair of incomparable points, then for each positive integer i, we let a_i denote the probability that $h_\lambda(y) - h_\lambda(x) = i$, and we let b_i denote the probability that $h_\lambda(x) - h_\lambda(y) = i$. We also set $b = b_1$ and let $B = \sum_i b_i = \text{Prob}\,(x > y)$. Then we set $\varepsilon = b/B$. Define the *height* of the pair $(\{a_i\}, \{b_i\})$ of sequences to be $\sum_i ia_i - \sum_i ib_i$; note that this is just $h(y) - h(x)$, the expected height difference.

We collect together various results from [13] in a lemma.

LEMMA 2.1.

$$a_1 = b_1 = b. \tag{2.1}$$

$$a_i = 0 \Rightarrow a_{i+1} = 0,\ \textit{for } i > 1, \quad \textit{and} \quad b_i = 0 \Rightarrow b_{i+1} = 0,\ \textit{for } i > 1. \tag{2.2}$$

$$\sum_{i \geqslant 1} a_i + \sum_{i \geqslant 1} b_i = 1. \tag{2.3}$$

$$a_2 + b_2 \leqslant a_1 + b_1. \tag{2.4}$$

$$a_{i+1} \leqslant a_i + a_{i+2},\ \textit{for } i \geqslant 2 \quad \textit{and} \quad b_{i+1} \leqslant b_i + b_{i+2},\ \textit{for } i \geqslant 2. \tag{2.5}$$

$$a_{i+1}^2 \geqslant a_i a_{i+2},\ \textit{for } i \geqslant 2 \quad \textit{and} \quad b_{i+1}^2 \geqslant b_i b_{i+2},\ \textit{for } i \geqslant 2. \tag{2.6}$$

Inequalities (2.1), (2.2) and (2.3) are trivial, but already (2.4) requires a clever little argument, and (2.5) is more substantial. A simpler proof of (2.5), based on a generalisation of (2.4), was provided by Felsner and Trotter [6]. The proof of (2.6) uses the Alexandrov–Fenchel inequalities for mixed volumes; a highly non-elementary piece of theory.

The basic approach of [13] may now be summarized as follows. Since P is not a chain, it follows that we may choose an ordered pair (x, y) with $0 \leqslant h(y) - h(x) < 1$; necessarily x and y are incomparable. Kahn and Saks prove that, if the sequences a_i and b_i satisfy (2.1)–(2.6), and the condition that the height of the pair of sequences be at most 1, then $B = \sum_{i \geqslant 1} b_i \geqslant 3/11$. (If we have $h(y) - h(x) < 1$, then we obtain the strict inequality $B > 3/11$.) Together with Lemma 2.1, this technical result implies Theorem 1.2.

We go a little deeper into the Kahn–Saks method here, for later use.

Say that a (two-way) sequence $(\{a_i\}_{i \geqslant 1}, \{b_i\}_{i \geqslant 1})$ of non-negative real numbers is *packed*, with parameters B, ε, k ($0 \leqslant B \leqslant 1/3$, $0 < \varepsilon \leqslant 1$, $k \in \mathbb{N}$), if it is of the form:

(1) $b_i = B\varepsilon(1 - \varepsilon)^{i-1}$ for all $i \geqslant 1$,

(2) $a_i = B\varepsilon(1 + \varepsilon)^{i-1}$ for $1 \leqslant i \leqslant k$,

(3) $a_{k+1} + a_{k+2} = 1 - \sum_{i \geqslant 1} b_i - \sum_{i=1}^{k} a_i$, $a_i = 0$ for $i \geqslant k + 2$, and either:

Case (i) $k \geqslant 2$, $a_{k+2} = 0$, with $\varepsilon/(1 + \varepsilon) \leqslant a_{k+1}/a_k \leqslant 1$, or

Case (ii) $a_{k+1} = a_k + a_{k+2}$, with $1 \leqslant a_{k+1}/a_k \leqslant 1 + \varepsilon$.

Note that packed sequences satisfy inequalities (2.1) through (2.6), and have $\sum b_i = B$. Note also that, for each pair (B, ε) there is exactly one packed sequence with parameters B, ε and k, for some k. The easiest way to see this is to set about constructing such a pair of sequence: B and ε determine the values of the b_i, and also that of $a_1 = b_1$. Then one must set $a_{j+1} = (1+\varepsilon)a_j$, until such time as there is not enough probability left to satisfy $a_{j+2} \geqslant \varepsilon a_j$; at this point, one sets a_{j+1} as large as possible. Either this will give $a_{j+1} \leqslant a_j$ and $a_{j+2} = 0$ (Case (i)), or $a_{j+1} \geqslant a_j$ and $a_{j+2} = a_{j+1} - a_j$ (Case (ii)). For each fixed $B \leqslant 1/3$, decreasing ε from 1 to 0 leads us through the cases in order: Case (ii), $k = 1$; Case (i), $k = 2$; Case (ii), $k = 2$; Case (i), $k = 3$; etc. The sequences are continuous functions of ε throughout this process.

Kahn and Saks calculated that, if we are in case (i) for some k, then

$$\frac{1+2\varepsilon+2\varepsilon^2}{1+\varepsilon} \leqslant \frac{(1+\varepsilon)^{1-k}}{B} \leqslant 1+2\varepsilon, \tag{2.7}$$

while if we are in Case (ii) for some k, then

$$1+2\varepsilon \leqslant \frac{(1+\varepsilon)^{1-k}}{B} \leqslant 1+2\varepsilon+2\varepsilon^2. \tag{2.8}$$

The main interest in packed sequences stems from the following lemma, also taken from Kahn and Saks [13].

LEMMA 2.2. *Suppose that* $(\{a_i\}, \{b_i\})$ *is any two-way sequence satisfying inequalities* (2.1) *through* (2.6), *with* $\sum b_i = B$, *and* $b_1/B = \varepsilon$. *Then there is a packed sequence with the same values of* B *and* ε, *and height no greater than that of* $(\{a_i\}, \{b_i\})$.

Now we define $H(B, \varepsilon)$ to be the height of the unique packed sequence with parameters B and ε, for $0 \leqslant B \leqslant 1/3$ and $0 < \varepsilon \leqslant 1$. Note that, for each fixed B, this is a continuous function of ε. Kahn and Saks proved that $H(3/11, \varepsilon) \geqslant H(3/11, 1) = 1$ for every ε, which, combined with Lemma 2.2, gives their result.

The following are expressions for $H(B, \varepsilon)$ in the various ranges, essentially taken from Kahn and Saks.

$$\text{Case (i): } H(B,\varepsilon) = k+1-\frac{B}{\varepsilon}(1+\varepsilon)^{k+1}. \tag{2.9}$$

$$\text{Case (ii): } H(B,\varepsilon) = k+\frac{3}{2}-\frac{B}{2\varepsilon}(1+\varepsilon)^{k-1}(4\varepsilon^2+5\varepsilon+2). \tag{2.10}$$

LEMMA 2.3. *For each fixed* B *with* $0 \leqslant B \leqslant 1/3$, *the function* $H(B, \varepsilon)$ *is decreasing in* ε, *except in case* (ii), *with* $k = 1$, *when* $\varepsilon \leqslant 1/\sqrt{2}$.

Note that the exceptional range only occurs when $B \geqslant 1 - 1/\sqrt{2} \simeq 0.293$. The range is at its largest when $B = 1/3$, when it is $[0.618\ldots, 0.707\ldots]$. Note also that, in Case (ii), $k = 1$, we have $H(B,\varepsilon) = H(B, 1/2\varepsilon)$, so, since $\varepsilon \leqslant 1/2$ is never a possibility, we have that $H(B,\varepsilon) \leqslant H(B,1)$ for every ε, for any $B \leqslant 1/3$.

Proof. Again, much of this is contained in Kahn and Saks [13]. Because of the continuity, we need only prove that the function is decreasing in each range.

Case (i) is relatively straightforward. First we claculate that

$$\frac{\partial H(B,\varepsilon)}{\partial \varepsilon} = \frac{B(1+\varepsilon)^k}{\varepsilon^2}(1-\varepsilon k). \tag{2.11}$$

Now if $\varepsilon < 1/k$, then

$$3 \leqslant \frac{1}{B} < (1+\varepsilon)^{k-1}(1+2\varepsilon) < \left(1+\frac{1}{k}\right)^{k-1}\left(1+\frac{2}{k}\right), \tag{2.12}$$

which is easily seen to be false for all integers $k \geqslant 2$. Therefore the derivative is negative for every range where Case (i) applies.

Case (ii) is a little more delicate. Here we have:

$$\frac{\partial H(B,\varepsilon)}{\partial \varepsilon} = \frac{B(1+\varepsilon)^{k-2}}{2\varepsilon^2}\Big[(1+\varepsilon)(2-4\varepsilon^2) - \varepsilon(4\varepsilon^2+5\varepsilon+2)(k-1)\Big]. \tag{2.13}$$

For $k = 1$, this is positive just when $\varepsilon^2 < 1/2$, as claimed.

For $k = 2$, one needs to verify directly that the condition

$$(1+\varepsilon)^{k-1} \geqslant \frac{1}{B(1+2\varepsilon+2\varepsilon^2)} \geqslant \frac{3}{1+2\varepsilon+2\varepsilon^2} \tag{2.14}$$

implies that $\varepsilon > 0.4044$, and that this in turn implies that the derivative is negative.

For $k \geqslant 3$, the bound in (2.14) implies that $\varepsilon > 3/(3k+1.1)$, which in turn implies that the derivative is negative. □

Lemma 2.2 tells us that, if our aim is to maximise $h(y) - h(x)$ subject to constraints on B and possibly also ε, then we may restrict attention to packed sequences. Lemma 2.3, and the remarks after the statement, say that we should take $\varepsilon = 1$ wherever possible. Thus, in the case $B = 3/11$, the extremal sequence is given by: $b_1 = a_1 = 3/11$, $a_2 = 4/11$ and $a_3 = 1/11$, the packed sequence with $\varepsilon = 1$, satisfying Case (ii) with $k = 1$.

The Kahn–Saks proof actually gives the stronger result that, for any pair x, y of incomparable elements of P such that $0 \leqslant h(y) - h(x) \leqslant 1$, we have $\mathrm{Prob}(x > y) \geqslant 3/11$. This stronger result is best possible, as there is a six point poset (see

Trotter [20], for example) containing an incomparable pair x, y with $h(y) - h(x) = 1$ and $\text{Prob}\,(x > y) = 3/11$.

As Kahn and Saks point out in [13], one way to improve the constant in Theorem 1.2 would be to show that there exists a positive absolute constant γ so that if P is not a chain, then it is always possible to find an ordered pair (x, y) with $0 \leqslant h(y) - h(x) \leqslant 1 - \gamma$. However, nobody has yet been able to settle whether such a γ exists. If it does, then as shown by Saks in [18], it must satisfy $\gamma \leqslant 0.133$. Even this value would not be enough to prove $\delta(P) \geqslant 1/3$. Our methods show that it would imply $\delta(P) \geqslant (3 + 2\gamma)/11 \simeq 0.297$. In fact, as noted by Felsner and Trotter in [6], if one is going to argue solely on the basis of packed sequences, only when $0 \leqslant h(y) - h(x) \leqslant 2/3$ can we safely conclude that $1/3 \leqslant \text{Prob}\,(x > y) \leqslant 2/3$. So we need a new idea.

Our approach towards improving Theorem 1.3 is to look at the relative heights of *three* elements rather than two. We prove the following result. Here and throughout, $a \| b$ means that a and b are incomparable.

THEOREM 2.4. *Let x, y, z be distinct points in a finite poset P, not forming a chain $x < y < z$. Suppose that $h(x) \leqslant h(y) \leqslant h(z) \leqslant h(x) + 2$.*

(i) *If $x < y$ in P, then* $\text{Prob}\,(y > z) \geqslant 1/3$.

(ii) *If $y < z$ in P, then* $\text{Prob}\,(x > y) \geqslant 1/3$.

(iii) *If $x \| y$ and $y \| z$ in P, then either*

$$\text{Prob}\,(x > y) \geqslant 1/3, \quad \textit{or} \quad \text{Prob}\,(y > z) \geqslant 1/3, \quad \textit{or}$$
$$\text{Prob}\,(x > y) + \text{Prob}\,(y > z) \geqslant (5 - \sqrt{5})/5.$$

We also have the following simple result.

LEMMA 2.5. *For any finite poset P, not a chain, with at least three elements, there are three distinct elements x, y, z, not forming a chain $x < y < z$, such that $h(x) \leqslant h(y) \leqslant h(z) \leqslant h(x) + 2$.*

Proof. Clearly we may assume that P contains no element comparable to all others. This implies that, if $x < y$ in P, then $h(y) - h(x) > 1$. Therefore if x, y, z satisfy $h(x) \leqslant h(y) \leqslant h(z) \leqslant h(x) + 2$, we cannot have $x < y < z$ in P.

Write the elements of P in non-decreasing order of average height, as $x_1, \ldots, x_n$. Note that $h(x_1) + h(x_2) \geqslant 3$, and $h(x_n) + h(x_{n-1}) \leqslant 2n - 1$. If n is odd, we deduce that $[h(x_n) - h(x_1)] + [h(x_{n-1}) - h(x_2)] \leqslant 2n - 4$, so either $h(x_n) - h(x_1) \leqslant n - 1$, or $h(x_{n-1}) - h(x_2) \leqslant n - 3$. In either case, we can write the difference as a sum of terms of the form $h(x_{k+2}) - h(x_k)$, and so find a k such that $h(x_{k+2}) - h(x_k) \leqslant 2$, as required. The proof for the case n even is similar. □

Theorem 2.4 and Lemma 2.5 clearly imply Theorem 1.3. Our primary task is thus to prove Theorem 2.4. In the next section, we develop a new inequality, similar in theme to inequality (2.6). Then we use this, together with other inequalities, to derive Theorem 2.4.

3. The Cross Product Conjecture

Let P be a finite poset, and let x, y, z be distinct elements of P. For $i, j \geqslant 1$, set $L(i, j)$ equal to the number of linear extensions λ of P in which $h_\lambda(y) - h_\lambda(x) = i$ and $h_\lambda(z) - h_\lambda(y) = j$. Also set $p(i, j) = L(i, j)/L(P)$, the probability that $h_\lambda(y) - h_\lambda(x) = i$ and $h_\lambda(z) - h_\lambda(y) = j$.

We make the following conjecture.

CONJECTURE 3.1 (The Cross Product Conjecture). *For any finite poset P and any integers $i, j \geqslant 1$,*

$$L(i, j)L(i + 1, j + 1) \leqslant L(i, j + 1)L(i + 1, j).$$

We have not been able to settle the Cross Product Conjecture, but we have been able to prove the following special case – and this is enough for the results of this paper.

THEOREM 3.2. *For any finite poset P,*

$$L(1, 1)L(2, 2) \leqslant L(1, 2)L(2, 1),$$

and therefore

$$p(1, 1)p(2, 2) \leqslant p(1, 2)p(2, 1).$$

Before proceeding with the proof of Theorem 3.2, we comment that this is the only new non-linear inequality we need for the proof of Theorem 1.3. Thus Theorem 3.2 plays the same role for us as the log-concavity statement (2.6) does for Kahn and Saks in [13]. Just as for (2.6), the proof of Theorem 3.2 is based on a powerful combinatorial tool that has been found useful in various similar contexts – the Ahlswede–Daykin Four Functions Theorem [1]. This result can be stated (not quite in full generality) as follows.

THEOREM 3.3. *Let $\mathcal{L}$ be a finite distributive lattice, and let $\alpha, \beta, \gamma, \delta$ be four functions from $\mathcal{L}$ to the positive reals satisfying:*

$$\alpha(A)\beta(B) \leqslant \gamma(A \vee B)\delta(A \wedge B)$$

for any $A, B \in \mathcal{L}$. Then

$$\sum_{A\in\mathcal{L}} \alpha(A) \sum_{A\in\mathcal{L}} \beta(A) \leqslant \sum_{A\in\mathcal{L}} \gamma(A) \sum_{A\in\mathcal{L}} \delta(A).$$

Throughout this section, we will be dealing with restrictions of the poset P to subsets of its ground-set. We shall abuse terminology slightly by referring to the restriction to a subset X as simply X, so in particular we write $L(X)$ for the number of linear extensions of the poset induced by P on a subset X of the ground set. Also, we refer to $L(X)$ as the number of linear extensions of X.

We shall also make use of the following consequence of the Four Functions Theorem. This result was first proved by Fishburn [7], and a simpler proof, based on an inequality of Shepp [19], was supplied by Brightwell [2]. See also [4] for further applications.

THEOREM 3.4. *Let P be a finite poset, and suppose that V and W are two up-sets in P. Then*

$$\frac{L(V)\,L(W)}{L(V\cup W)\,L(V\cap W)} \leqslant \frac{|V|!\,|W|!}{|V\cup W|!\,|V\cap W|!} \leqslant 1.$$

In this note, we do not need the full strength of Fishburn's Inequality, Theorem 3.4: we use only that, with V and W as above, $L(V)\,L(W) \leqslant L(V\cup W)\,L(V\cap W)$. We note next one fairly straightforward consequence. For a finite poset P, a minimal element x of P, and an integer $i \geqslant 1$, set $S_i(P;x)$ equal to the number of linear extensions λ of P in which $h_\lambda(x) = i$.

LEMMA 3.5. *Suppose x is a minimal element in a poset Y, and Z is an up-set of Y containing x. Then*

$$S_2(Z;x)/S_1(Z;x) \leqslant S_2(Y;x)/S_1(Y;x).$$

Proof. Note that $S_1(Z;x)$ is just the number of linear extensions of $Z\setminus\{x\}$, and $S_2(Z;x)$ is the number of such linear extensions in which the bottom element is incomparable with x. Hence $S_2(Z;x)/S_1(Z;x)$ is the probability that a randomly chosen linear extension of $Z\setminus\{x\}$ has its bottom element incomparable to x.

Let $V(Z)$ be the set of elements v of Z such that x is the only element below v in Z. Then we have

$$\begin{aligned}\frac{S_2(Z;x)}{S_1(Z;x)} &= 1 - \sum_{v\in V(Z)} \text{Prob}\,(v \text{ is the bottom element in a linear extension of } Z\setminus\{x\})\\ &= 1 - \sum_{v\in V(Z)} \frac{L(Z\setminus\{v,x\})}{L(Z\setminus\{x\})}.\end{aligned}$$

The same is true with Y in place of Z, so it is sufficient to prove that

$$\sum_{v\in V(Z)} \frac{L(Z\setminus\{v,x\})}{L(Z\setminus\{x\})} \geqslant \sum_{v\in V(Y)} \frac{L(Y\setminus\{v,x\})}{L(Y\setminus\{x\})}.$$

Now $V(Y) \subseteq V(Z)$, so it is sufficient to prove that, for any $v \in V(Y)$,

$$\frac{L(Z \setminus \{v,x\})}{L(Z \setminus \{x\})} \geqslant \frac{L(Y \setminus \{v,x\})}{L(Y \setminus \{x\})}.$$

This last inequality follows from Fishburn's Inequality, Theorem 3.4, with $V = Z \setminus \{x\}$ and $W = Y \setminus \{v,x\}$. □

Proof of Theorem 3.2. First note that we may as well suppose that $x < y < z$ in P, since we are counting only linear extensions where these relations are satisfied. Also, we may assume that y is the only element between x and z, since otherwise $L(1,1) = 0$.

Now set D equal to the set of elements of P below y, U equal to the set of elements above y, and I equal to the set of elements incomparable to y. So (D,U,I) is a partition of $P \setminus \{y\}$. Set $\mathcal{U}$ equal to the set of up-sets of I, and note that $\mathcal{U}$ is a distributive lattice under set-inclusion. For $A \in \mathcal{U}$, let A^c be the complement $I \setminus A$.

Observe that $L(i,j)$ is equal to the sum, over all elements A of $\mathcal{U}$, of the number of linear extensions λ of P in which: (i) $h_\lambda(y) - h_\lambda(x) = i$ and $h_\lambda(z) - h_\lambda(y) = j$, and (ii) $A^c < y < A$. This number is just the product of (i) the number $f_j(A)$ of linear extensions μ of $U \cup A$ in which $h_\mu(z) = j$, and (ii) the number $g_i(A^c)$ of linear extensions ν of $D \cup A^c$ in which $h_\nu(x) = |D \cup A^c| - i + 1$, i.e., exactly $i - 1$ elements come above x. Thus we have

$$L(i,j) = \sum_{A \in \mathcal{U}} f_j(A) g_i(A^c).$$

Our aim is to apply the Four Functions Inequality, Theorem 3.2, to the lattice $\mathcal{U}$ with:

$$\alpha(A) = f_1(A) g_1(A^c),$$
$$\beta(A) = f_2(A) g_2(A^c),$$
$$\gamma(A) = f_2(A) g_1(A^c),$$
$$\delta(A) = f_1(A) g_2(A^c).$$

This will imply our result, provided we can prove that the condition of the Four Functions Theorem is satisfied.

Thus it suffices to show that, for any $A, B \in \mathcal{U}$,

$$f_1(A) g_1(A^c) f_2(B) g_2(B^c) \leqslant f_2(A \cup B) g_1(A^c \cap B^c) f_1(A \cap B) g_2(A^c \cup B^c). \qquad (*)$$

In fact we shall prove that

$$f_1(A) f_2(B) \leqslant f_2(A \cup B) f_1(A \cap B); \qquad (**)$$

the analogous inequality for the g_j follows by symmetry, and $(*)$ then follows.

Inequality $(**)$ is trivial if either $f_1(A)$ or $f_2(B)$ is equal to 0, which is the case whenever either (i) A contains any element below z, or (ii) B contains more than one such element. We break the argument into two cases, depending on whether B does or does not contain an element $v < z$.

First we suppose that B does contain such a v. Now observe that $f_1(A)$, the number of linear extensions of $U \cup A$ with bottom element z, is simply equal to $L(U \cup A \setminus \{z\})$. Similarly $f_2(B) = L(U \cup B \setminus \{z, v\})$, and similarly for the other expressions in $(**)$. Thus $(**)$ follows in this case on applying Fishburn's Inequality with $V = U \cup A \setminus \{z\}$ and $W = U \cup B \setminus \{z, v\}$.

Now we move on to the other case, where z is minimal among the elements of $U \cup A \cup B$. In this case, Lemma 3.5 is applicable, and we observe that $f_i(C) = S_i(U \cup C; z)$, for any i, and any set $C \subset I$ such that z is minimal in $U \cup C$. Thus in particular we have

$$f_2(B)/f_1(B) \leqslant f_2(A \cup B)/f_1(A \cup B).$$

Also, Fishburn's Inequality tells us that

$$f_1(A)f_1(B) \leqslant f_1(A \cup B)f_1(A \cap B).$$

Combining these two inequalities gives us inequality $(**)$ in this case as well, which completes the proof. □

Perhaps the methods of this section can be extended to prove that, for any for any finite poset P and any positive integers i, j,

$$L(1,1)L(i,j) \leqslant L(1,j)L(i,1),$$

but something more powerful seems to be needed to prove the general form of the Cross Product Conjecture.

4. Theorem 2.4 – Two Easy Cases

Suppose we have three points x, y and z of our poset P, not forming a 3-element chain, with $h(x) \leqslant h(y) \leqslant h(z) \leqslant 2 + h(x)$. We break the proof of Theorem 2.4 into cases, depending on the subposet of P formed by $\{x, y, z\}$. Taking advantage of duality, we observe that we need only consider the following four situations.

Case A: $x < z$ and $y < z$ in P.

Case B: $y < z$, $x \| y$ and $x \| z$ in P.

Case C: $\{x, y, z\}$ is a 3-element antichain.

Case D: $x < z$, $x \| y$ and $y \| z$ in P.

Theorem 2.4 then says that, if Cases A or B hold, then Prob $(x > y) \geqslant 1/3$, and if Cases C or D hold, then either Prob $(x > y) \geqslant 1/3$, or Prob $(y > z) \geqslant 1/3$, or Prob $(x > y) +$ Prob $(y > z) \geqslant (5 - \sqrt{5})/5$.

In arguments to follow, we will continue to use the previous definitions for b, B, a_i, b_i and ε; the pair of elements they are defined for (usually (x, y)) will be clear from the context. Later, we will we need to consider sequences for two pairs simultaneously – then we will clarify this in the notation.

In the remainder of this section, we deal with Cases A and B. We begin with Case A.

THEOREM 4.1. *If Case A holds, then*

$$\text{Prob}\,(x < y) \leqslant 2/3.$$

Proof. In this case, note that $h_\lambda(z) - h_\lambda(x) \geqslant 1$ for every linear extension λ, that Prob $(h_\lambda(z) - h_\lambda(x)) = 1 \leqslant$ Prob $(x > y) = B$, and that

$$\text{Prob}\,(h_\lambda(z) - h_\lambda(x) \leqslant 2) \leqslant \text{Prob}\,(x > y) + \text{Prob}\,(h_\lambda(y) - h_\lambda(x) = 1) = B + b.$$

Thus we have

$$2 \geqslant h(z) - h(x) \geqslant B + 2b + 3(1 - B - b).$$

This implies that $1 \leqslant 2B + b \leqslant 3B$ so that $B \geqslant 1/3$. □

Incidentally, Theorem 4.1 is best possible, as shown by a poset on 4 elements x, y, z, w, with $x < z$ and $w < y < z$.

THEOREM 4.2. *If Case B holds, then*

$$\text{Prob}\,(x < y) \leqslant 2/3.$$

Proof. Obviously,

$$h(z) - h(y) = 1 + \sum_{i,j}(j-1)p(i,j) \geqslant 1 + p(-1,2) + 2\sum_{i\geqslant 3} p(-1,i) + \sum_{i \geqslant 2} p(1,i).$$

Using $\sum_{i\geqslant 1} p(1,i) = \sum_{i\geqslant 2} p(-1,j) = b_1$ and $p(-1,2) = p(1,1) = p(-2,1) \leqslant b_2$ we obtain $h(z) - h(y) \geqslant 1 + b_2 + 2(b_1 - b_2) + (b_1 - b_2) = 1 + 3b_1 - 2b_2$.

This leads to a correlation between the height of x and y and their probability of being reversed and close to each other:

$$h(y) - h(x) \leqslant 2 - (h(z) - h(y)) \leqslant 1 - 3b_1 + 2b_2.$$

Now suppose that there are sequences $\{a_i\}_{i\geqslant 1}$ and $\{b_i\}_{i\geqslant 1}$ satisfying the conditions of (2.1)–(2.6), with $\sum_{i\geqslant 1} i(a_i - b_i) \leqslant 1 - 3b_1 + 2b_2$ and $\sum_{i\geqslant 1} b_i \leqslant B$. Then the packed sequences $a_i = b(1+\varepsilon)^i$ and $b_i = b(1-\varepsilon)^i$ also satisfy all these conditions. Therefore we can analyze the situation with the techniques of [13]. It turns out that the worst case occurs in Case (ii) with $k = 3$. For this value of k, it may be verified that $B \geqslant 0.335$, which is a little more than what is claimed in the statement of the theorem. □

5. Theorem 2.4 – Case D

In this section, we assume that we are in Case D, i.e., that x, y, z are three elements of P, with $h(x) \leqslant h(y) \leqslant h(z) \leqslant h(x) + 2$, $x < z$, $x \| y$ and $y \| z$ in P. Set $B = \text{Prob}\,(x > y)$, as before, and $B' = \text{Prob}\,(y > z)$. Our aim is to prove that $B + B' \geqslant (5 - \sqrt{5})/5$. We give reasonably full details of the computation in this case, since it is critical to our analysis.

We start by noting that, for any $j \geqslant 2$, $p(-1, j) = p(1, j-1)$, since swapping x and y gives a bijection between the two sets of linear extensions being counted, and similarly $p(j, -1) = p(j-1, 1)$. Note also that, since $x < z$ in P, $p(i, j) = 0$ whenever $i + j \leqslant 0$.

To simplify the computational efforts, we let $X = B + B'$, $x_1 = p(1, 1)$, $x_2 = p(1, 2) + p(2, 1)$, $x_3 = p(2, 2)$, $x_4 = p(1, 3) + p(3, 1)$, $x_5 = p(2, 3) + p(3, 2)$ and $x_6 = p(1, 4) + p(4, 1)$.

Our method is to produce various inequalities relating X and the x_i, and then to prove that, subject to the various inequalities, the minimum value of X is $(5 - \sqrt{5})/5$. The inequalities we derive and use may seem to be somewhat arbitrary; undoubtedly there are other inequalities, perhaps stronger and/or more natural, that can be derived. Motivation for the particular inequalities chosen came from two sources: (a) we know that, in the infinite poset Q defined in Section 7, we have $X = (5 - \sqrt{5})/5$, $x_1 = (3\sqrt{5} - 5)/10$, $x_2 = (10 - 4\sqrt{5})/5$, $x_3 = (7\sqrt{5} - 15)/10$, and $x_4 = x_5 = x_6 = 0$, so the inequalities we use should be tight for this assignment of values, (b) we carried out extensive numerical experiments using various computer algebra packages, and these suggested which inequalities would be useful. In particular, the Cross Product Conjecture was discovered with these experiments.

We now begin the derivation of the required inequalities. First we use the inequality $h(z) - h(x) \leqslant 2$ to obtain:

$$\begin{aligned}
2 &\geqslant \sum_{i,j} (i+j)p(i,j) \\
&\geqslant 2p(1,1) + 3p(1,2) + 3p(2,1) + 4p(1,3) + 4p(3,1) + 4p(2,2) + \\
&\quad + 5p(1,4) + 5p(4,1) + 5p(2,3) + 5p(3,2) + 6\big[1 - B - B' - \\
&\quad - p(1,1) - p(1,2) - p(2,1) - p(1,3) - p(3,1) - p(2,2) - \\
&\quad - p(1,4) - p(4,1) - p(2,3) - p(3,2)\big] + 2p(-1,3) + 2p(3,-1) + \\
&\quad + 3p(-1,4) + 3p(4,-1) + 4p(-1,5) + 4p(5,-1) + \big[B + B' - \\
&\quad - p(-1,3) - p(3,-1) - p(-1,4) - p(4,-1) - p(-1,5) - p(5,-1)\big] \\
&= 2x_1 + 3x_2 + 4x_4 + 4x_3 + 5x_6 + 5x_5 + 6(1 - X - x_1 - x_2 - x_4 - \\
&\quad - x_3 - x_6 - x_5) + 2x_2 + 3x_4 + 4x_6 + (X - x_2 - x_4 - x_6).
\end{aligned}$$

Rearranging, and noting that $x_6 \geqslant 0$, we obtain that:

$$4 \leqslant 5X + 4x_1 + 2x_2 + 2x_3 + x_5. \qquad (5.1)$$

Next we have that

$$B + B' \geqslant p(-1,2) + p(-1,3) + p(2,-1) + p(3,-1),$$

so

$$2x_1 + x_2 \leqslant X. \tag{5.2}$$

Our next inequality requires an easy lemma.

LEMMA 5.1.

$$p(2,3) \leqslant p(1,3) + p(1,4).$$

Proof. We give an injection from the set A of linear extensions counted by $L(2,3)$ to the union of the sets A_1 and A_2 of linear extensions counted by $L(1,3)$ and $L(1,4)$ respectively.

For a linear extension λ in A, there is exactly one element w with $x < w < y$ in λ. If $x\|w$ in P, then swap x and w to obtain a linear extension in A_1. If not, then $w\|y$ in P, so we may swap w and y to obtain a linear extension in A_2. The map described is clearly an injection. □

Similarly we have $p(3,2) \leqslant p(3,1) + p(4,1)$, and adding the two inequalities, we have:

$$x_5 \leqslant x_4 + x_6. \tag{5.3}$$

Since the sum of all probability is one, we have

$$X + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leqslant 1. \tag{5.4}$$

The final inequality we need comes from Theorem 3.2. Observe that $p(1,2) \times p(2,1) \leqslant (p(1,2) + p(2,1))^2/4$, so that

$$x_1 x_3 \leqslant x_2^2/4. \tag{5.5}$$

We claim that, subject to the inequalities (5.1)–(5.5), and the requirement that all the variables are non-negative, the minimum value of X is $(5-\sqrt{5})/5$.

First, adding (5.3) and (5.4) gives:

$$X + x_1 + x_2 + x_3 + 2x_5 \leqslant 1. \tag{5.6}$$

Now we derive two simpler inequalities from (5.1) and (5.6); first we take $2 \times (5.1) + (5.6)$ and get

$$7 \leqslant 9X + 7x_1 + 3x_2 + 3x_3, \tag{5.7}$$

then we take $4\times(5.6)+(5.1)$ and note that $x_5 \geqslant 0$, to obtain

$$2x_2 + 2x_3 \leqslant X. \tag{5.8}$$

Now we set $y_1 = x_1/X$, $y_2 = x_2/X$, $y_3 = x_3/X$ in (5.5), (5.2), (5.8) and (5.7), and find that

$$y_1 y_3 \leqslant y_2^2/4, \tag{5.9}$$

$$2y_1 + y_2 \leqslant 1, \tag{5.10}$$

$$2y_2 + 2y_3 \leqslant 1, \tag{5.11}$$

$$7 \leqslant X(9 + 7y_1 + 3y_2 + 3y_3). \tag{5.12}$$

To minimise X subject to (5.9)–(5.12) is equivalent to maximising $7y_1 + 3y_2 + 3y_3$ subject to (5.9)–(5.11). (Again, it is understood that all variables are non-negative.)

It is easy to see that, at the optimum, we have equality in (5.9). Indeed, if we do not, then it is possible to increase y_1 by some $\varepsilon > 0$, and decrease y_2 by 2ε, remaining feasible and increasing the objective. Also, we must have equality in (5.10), since otherwise we can increase y_2 by some $\varepsilon > 0$, and decrease y_3 by ε, keeping the objective fixed, remaining feasible, and breaking the equality in (5.9). Thus we may substitute $y_2 = 1 - 2y_1$ and

$$y_3 = \frac{(1-2y_1)^2}{4y_1} = \frac{1}{4y_1} - 1 + y_1$$

to reduce the problem to that of maximising $4y_1 + 3/(4y_1)$ subject to $-2y_1 + 1/(2y_1) \leqslant 1$ and $y_1 \leqslant 1/2$. The first constraint works out to $y_1 \geqslant (\sqrt{5}-1)/4$. The objective function is concave, so the maximum is obtained at one of the two endpoints of the range: it turns out to be larger at the lower end, where it takes the value $(7\sqrt{5}-1)/4$. Substituting back into (5.12) gives

$$7 \leqslant X\frac{35+7\sqrt{5}}{4}, \quad \text{i.e.,} \quad X \geqslant \frac{5-\sqrt{5}}{5},$$

as claimed.

6. Theorem 2.4 – Case C

The final case we have left to consider is Case C, where x, y, z form a three-element chain in P, with $h(x) \leqslant h(y) \leqslant h(z) \leqslant h(x) + 2$. For this case, we use the methods of Kahn and Saks [13], as set out in Section 2.

LEMMA 6.1. *If x and z are incomparable elements of a poset P, with $h(z) \leqslant h(x) + 2$, then* $\text{Prob}\,(x > z) \geqslant 3/22$.

Proof. This follows from Lemmas 2.2 and 2.3. Indeed, any sequence satisfying (2.1)–(2.6) with $B = 3/22$ has height at least that of the packed sequence with parameters $B = 3/22$ and $\varepsilon = 1$. This sequence is given by $b_1 = a_1 = 3/22$, $a_2 = 6/22$, $a_3 = 8/22$, $a_4 = 2/22$, with a height of 2. □

LEMMA 6.2. *If x, y, z are as given, then*

$$\text{Prob}\ (h_\lambda(x) - h_\lambda(y) \geqslant 2) + \text{Prob}\ (h_\lambda(y) - h_\lambda(z) \geqslant 2) \geqslant \frac{1}{11}.$$

Proof. Note that the number of linear extensions of P in which z, y, x occur consecutively in that increasing order is equal to the number with y, z, x consecutively in that order, and also equal to the number with z, x, y consecutively in that order. Thus the probability that z, y, x occur consecutively in that order is at most one third of the probability that $x > z$.

Now observe that, if a linear extension λ of P has $z < x$, but does not feature z, y, x consecutively in that order, then either $h_\lambda(x) - h_\lambda(y) \geqslant 2$ or $h_\lambda(y) - h_\lambda(z) \geqslant 2$ (or both). So we have

$$\begin{aligned}
&\text{Prob}\ (h_\lambda(x) - h_\lambda(y) \geqslant 2) + \text{Prob}\ (h_\lambda(y) - h_\lambda(z) \geqslant 2) \\
&\quad \geqslant \text{Prob}\ (h_\lambda(x) - h_\lambda(y) \geqslant 2 \text{ or } h_\lambda(y) - h_\lambda(z) \geqslant 2) \\
&\quad \geqslant \frac{2}{3}\text{Prob}\,(x > z) \geqslant \frac{1}{11},
\end{aligned}$$

with the final inequality following from Lemma 6.1. □

From now on, we consider the two pairs (x, y) and (y, z). We will continue to use the notation a_i to denote $\text{Prob}\,(h_\lambda(y) - h_\lambda(x) = i)$, and similarly for b_i; we introduce the notation $a'_i = \text{Prob}\,(h_\lambda(z) - h_\lambda(y) = i)$, and $b'_i = \text{Prob}\,(h_\lambda(y) - h_\lambda(y) = z)$. We set, as before, $B = \sum b_i$ and $\varepsilon = b_1/B$; we define also $B' = \sum b'_i$ and $\varepsilon' = b'_1/B'$. Lemma 6.2 tells us that

$$B(1 - \varepsilon) + B'(1 - \varepsilon') \geqslant \frac{1}{11}. \tag{6.1}$$

Suppose that $B + B' = \text{Prob}\,(x > y) + \text{Prob}\,(y > z) = (5 - \sqrt{5})/5$.

Our plan is to incorporate the extra constraint (6.1) into the Kahn–Saks analysis for the sequences corresponding to the two pairs, and deduce that the heights of $(\{a_i\}, \{b_i\})$ and $(\{a'_i\}, \{b'_i\})$ sum to more than 2. To this end, Lemma 2.2 tells us that we may assume the sequences are packed. The following lemma thus implies the result for Case C.

LEMMA 6.3. *If* $0 < B \leqslant B' \leqslant 1/3$, $B + B' = (5 - \sqrt{5})/5$, *and* $B(1-\varepsilon) + B'(1-\varepsilon') \geqslant 1/11$, *then* $H(B,\varepsilon) + H(B',\varepsilon') > 2$.

Proof. We break the analysis into three cases.

(a) First, let us assume that $B \geqslant 3/10$, and that $1 + 2\varepsilon + 2\varepsilon^2 \leqslant 10/3$, which implies that $\varepsilon < 0.7 < 1/\sqrt{2}$. By Lemma 2.3, we have $H(B,\varepsilon) + H(B',\varepsilon') \geqslant H(B,\varepsilon_0) + H(B',1)$, where $1/B = 1 + 2\varepsilon_0 + 2\varepsilon_0^2$. Note that $0.618 \simeq (\sqrt{5}-1)/2 \leqslant \varepsilon_0 \leqslant 0.7$.

For (B,ε) satisfying Case (ii), $k = 1$, we have

$$H(B,\varepsilon) = \frac{5}{2} - \frac{B}{2\varepsilon}(4\varepsilon^2 + 5\varepsilon + 2)$$
$$= \frac{5 - 11B}{2} + B(1-\varepsilon) - B\frac{(1-\varepsilon)^2}{\varepsilon}.$$

Thus we obtain

$$H(B,\varepsilon_0) + H(B',1) = 5 - \frac{11}{2}(B + B') + B\frac{(1-\varepsilon_0)(2\varepsilon_0 - 1)}{\varepsilon_0}$$
$$= 5 - \frac{11}{2}\frac{5-\sqrt{5}}{5} + \frac{(1-\varepsilon_0)(2\varepsilon_0 - 1)}{\varepsilon_0(1 + 2\varepsilon_0 + 2\varepsilon_0^2)}.$$

This function is minimised in the range $[0.618\ldots, 0.7]$ at the lower endpoint, when it is equal to $(10 + 9\sqrt{5})/15 \simeq 2.0083$.

(b) Our second case is where B is any value in the feasible range $[0.219\ldots, 1/3]$, and $(1 + 2\varepsilon)(1 + \varepsilon) \leqslant 1/B$, i.e., ε is at or below the lower boundary of Case (i) with $k = 2$. Then, by Lemma 2.3, $H(B,\varepsilon) + H(B',\varepsilon') \geqslant H(B,\varepsilon_1) + H(B',1)$, where $(1 + 2\varepsilon_1)(1 + \varepsilon_1) = 1/B$, so certainly $\varepsilon_1 \geqslant 1/2$.

Now

$$H(B,\varepsilon_1) - H(B,1) = \frac{1}{2} - B\left(\frac{(1+\varepsilon_1)^3}{\varepsilon_1} - \frac{11}{2}\right)$$
$$= \frac{1}{2} - \frac{2 - 5\varepsilon_1 + 6\varepsilon_1^2 + 2\varepsilon_1^3}{2\varepsilon_1(1+\varepsilon_1)(1+2\varepsilon_1)},$$

and this expression is at least $1/12$ in the range $1/2 \leqslant \varepsilon_1 \leqslant 1$. Since $H(B,1) + H(B',1) = 5 - 11(5 - \sqrt{5})/5 \simeq 1.960$, we are done here.

(c) The remaining case is when neither the (B,ε) sequence nor the (B',ε') sequence fall into one of the above two cases. Note that both sequences must then satisfy either Case (ii) with $k = 1$ or Case (i) with $k = 2$.

Our aim is to show that the quantity

$$M(B,\varepsilon) = \frac{H(B,\varepsilon) - H(B,1)}{B(1-\varepsilon)}$$

is at least $1/2$. This will then imply that

$$H(B,\varepsilon) + H(B',\varepsilon') \geqslant H(B,1) + H(B',1) + \frac{1}{2}\left(B(1-\varepsilon) + B'(1-\varepsilon')\right)$$
$$\geqslant 5 - 11\left(\frac{5-\sqrt{5}}{5}\right) + \frac{1}{22} \simeq 2.0051,$$

which will complete the proof.

Suppose first that the (B,ε) sequence comes under Case (ii), $k=1$. If $\varepsilon \leqslant 0.69$, then $B \geqslant 1/(1+2\varepsilon+2\varepsilon^2) > 0.3$, and the (B,ε) sequence was covered by Case (a). Thus $\varepsilon \geqslant 0.69$. Then, from an earlier calculation, we have that

$$H(B,\varepsilon) = H(B,1) + B(1-\varepsilon) - B\frac{(1-\varepsilon)^2}{\varepsilon},$$

so $M(B,\varepsilon) = 2 - 1/\varepsilon \geqslant 2 - 1/0.69 > 1/2$, as desired.

Now suppose that the (B,ε) sequence comes under Case (i), $k=2$. If $B \geqslant 3/10$, then the sequence again comes under (a) above, so we may assume that $B \leqslant 3/10$. Now

$$M(B,\varepsilon) = \frac{3 - B(1+\varepsilon)^3/\varepsilon - (5-11B)/2}{B(1-\varepsilon)}$$
$$= \frac{1}{2B(1-\varepsilon)} - \frac{(1+\varepsilon)^3/\varepsilon - 11/2}{1-\varepsilon}.$$

Thus, for each fixed ε, $M(B,\varepsilon)$ is decreasing in B. So, to prove that $M(B,\varepsilon)$ is at least $1/2$, we may assume that B is as large as possible. Increasing B, remaining inside the case, we arrive at a point where either (i) $B = 1/(1+2\varepsilon+2\varepsilon^2)$, and $\varepsilon \geqslant 0.69$, which is on the boundary between the two cases, so we know that $M(B,\varepsilon) > 1/2$, or (ii) $B = 3/10$, and $\varepsilon \leqslant 0.7$, when one may verify directly that $M(3/10,\varepsilon) > 1/2$.

This completes the proof. □

We have now completed the proof of Theorem 2.4, and hence of Theorem 1.3. The calculation in this case is not designed to give the best possible bounds, and it seems to be just a matter of luck that the extra bound (6.1) gives enough improvement. We suspect that, if x, y, z are as given, then one of Prob$(x > y)$ and Prob$(y > z)$ will be significantly larger than $(5-\sqrt{5})/10 \simeq 0.2764$. An infinite example in [2] shows that they may both be as small as $(13+\sqrt{17})/68 \simeq 0.3106$.

7. Extension to the Infinite Case

In this section, we discuss in greater detail a class of partially ordered sets for which there is a natural way to extend the definition of Prob(x, y) when the ground set is infinite. For this class, the $1/3$–$2/3$ conjecture *fails*. But our Theorem 1.3 remains valid *and* is best possible.

Recall that a poset P is *thin* if there is some fixed k such that every element is incomparable with at most k others. It is also convenient to impose the condition that P be *locally finite*, i.e., that each set $\{z\colon\ x < z < y\}$, for $x, y \in P$, is finite. These conditions imply that P is countable, and that, for any pair x, y of elements, the number of elements between x and y in a linear extension of P is bounded.

Let P be an infinite, thin, locally finite poset on a ground-set X, containing elements x, y, z, and let $(X_n)_{n=1}^{\infty}$ be a sequence of finite subsets of X satisfying: (1) if $u, v \in X_n$ and $u < w < v$ in P, then $w \in X_n$, (2) $X_i \subseteq X_j$ for $i \leqslant j$, (3) $\bigcup_{n=1}^{\infty} X_n = X$. For $n \in \mathbb{N}$, let P_n be the partial order obtained by restricting P to X_n. It is proved by Brightwell [2] that, for any event $\mathcal{A}$ depending only on finitely many basic events of the form $a < b$, the probability of $\mathcal{A}$ in P_n converges to a limit, which is by definition the probability of $\mathcal{A}$ in P.

We fix an infinite, thin, locally finite poset P, and a sequence of subposets P_n with ground-sets X_n, as above. We clearly cannot define the average height of an element, but we can define the *average height difference* $h(x, y)$ of two elements x and y to be the limit of $h(y) - h(x)$ in P_n, as $n \to \infty$. It was proved in [2] that there necessarily exists a pair x, y with $0 \leqslant h(x, y) \leqslant 1$, and a similar proof establishes that there is a triple x, y, z with $h(x, y) \geqslant 0$, $h(y, z) \geqslant 0$, and $h(x, z) \leqslant 2$.

Fix such a triple x, y, z in P and, for n sufficiently large that X_n contains x, y and z, let $p_n(i, j)$ be the probability that $h_\lambda(y) - h_\lambda(x) = i$ and $h_\lambda(z) - h_\lambda(y) = j$ in a random linear extension λ of P_n. Then each $p_n(i, j)$ tends to a limit $p(i, j)$, which is the corresponding probability in P. Furthermore, if we have an inequality relating various of the $p(i, j)$, and perhaps also some average height differences, that is valid for all finite posets, then the inequality will carry over to the limit. In particular, Theorem 3.1 is valid, and so are all the other inequalities used in Sections 4–6 to derive the bounds on $B + B'$.

Thus all our proofs carry over into the infinite case, and Theorem 2.4, suitably restated in terms of average height differences, is valid for infinite, thin, locally finite posets. Thus $\delta(P) \leqslant (5 - \sqrt{5})/10$ for every locally finite thin poset P.

Finally, we can remove the condition of local finiteness, and obtain Theorem 1.4, readily enough, as follows. Every thin poset P has the structure of a family $(P_i)_{i \in I}$ of locally finite thin posets, indexed by a totally ordered set I, such that if $i < j$ in I, then every element of P_i is below every element of P_j in P. Then Prob$(x < y)$ in P, for $x \| y$, is defined as Prob$(x < y)$ in the P_i containing x and y. If P is not a chain, then one of the P_i is not a chain, and we may find a balancing pair in that poset.

8. Application to Sorting

The original motivation for studying balancing pairs in posets was the connection with sorting. Suppose we are to find an unknown linear extension of a finite partially ordered set P by making comparisons of pairs of elements of P. Thus, at each stage, we may choose some pair of elements x and y, and ask whether $x < y$ in the unknown linear extension. Let $S(P)$ denote the number of rounds required to find the linear extension, in the worst case. Thus $S(P)$ is the number of comparisons required to sort the elements, starting from the knowledge of the relations given by P. The fundamental problem was to answer whether $S(P) = O(\log L(P))$, i.e., is it always possible to determine an unknown linear extension of P with $O(\log L(P))$ rounds (questions). Theorem 1.2 implies a positive answer to this question (as does any result that shows $\delta_0 > 0$).

Indeed, at each step one can choose a pair (x, y) such that $3/11 \leqslant \text{Prob}\,(x > y) \leqslant 8/11$, and ask whether x is above y. Whatever the answer, the number of possible linear extensions is reduced by a factor of at most $8/11$ in this round. Therefore the number of rounds required to identify the linear extension is at most $-\log L(P)/\log 8/11$. Thus $\log L(P)/\log(11/8) \simeq 3.140 \log L(P)$ rounds suffice, and Theorem 1.3 improves this to $\log L(P)/\log((5-\sqrt{5})/2) \simeq 3.091 \log L(P)$. Later in this section, we will strengthen this a bit more.

None of the arguments in [12–14] or this paper yields an efficient algorithm for the original sorting problem, since they do not provide an efficient method for determing how to locate the balancing pair. In [11], Kahn and J. Kim have taken a totally different approach to the sorting problem. Using a concept of entropy for posets, they show the existence of a polynomial time algorithm for sorting in $O(\log L(P))$ rounds. Their algorithm shows how to efficiently locate pairs to use in queries so that, regardless of the responses, the determination of the unknown linear extension is made in $O(\log L(P))$ rounds. However, at individual rounds, the pairs need not be balanced in the sense that for a given pair (x, y) used in the algorithm, $\text{Prob}\,(x > y)$ may be arbitrarily close to zero. We have already seen that $S(P) \leqslant -\log L(P)/\log(1-\delta_0)$ for every poset P, and elementary information theory gives us that $S(P) \geqslant \log L(P)/\log 2$ for every P. We define ϕ_0 to be the supremum, over all finite posets P, of $S(P)/\log L(P)$. From our previous remarks, we know $\phi_0 \leqslant 1/\log((5-\sqrt{5})/2) \simeq 3.091$. So the remainder of this section is aimed at improving this further, and also to give a lower bound for ϕ_0.

A better upper bound follows in a straightforward manner from a slightly closer look at Theorem 2.4 and Lemma 2.5. But first we point out that Fredman [9] has proved that $S(P) \leqslant \log L(P)/\log 2 + 2|P|$, so we are really concerned here with posets that are "almost sorted", i.e., that have rather few linear extensions compared with $|P|\,!$.

THEOREM 8.1. $\phi_0 \leqslant 4/\log 5 \simeq 2.485$.

Proof. We have to prove that $S(P) \leqslant 4 \log L(P)/\log 5$ for every finite poset P. Suppose P is a counterexample minimising (say) $|P| + |L(P)|$. We note that P contains no element comparable with all others, since otherwise P breaks up into smaller posets that can be sorted separately. The result is also true for the 2-element antichain, so we have that P is not a chain, and has at least three elements. So we may apply Lemma 2.5, and find three elements x, y, z, not forming a chain in P, such that $h(x) \leqslant h(y) \leqslant h(z) \leqslant h(x) + 2$.

Note that, since $h(x) \leqslant h(y)$, we certainly have $\mathrm{Prob}\,(x > y) \leqslant 2/3$. If also $\mathrm{Prob}\,(x > y) \geqslant 1/3$, then we compare x and y. Whatever the result of the comparison, we obtain a new poset P' with $L(P') \leqslant \frac{2}{3}L(P) \leqslant 5^{-1/4}L(P)$. By definition of P, we have that $S(P') \leqslant 4 \log L(P')/\log 5 \leqslant 4 \log L(P)/\log 5 - 1$, and so $S(P) \leqslant 4 \log L(P)/\log 5$, a contradiction.

Thus $\mathrm{Prob}\,(x > y) < 1/3$, and similarly $\mathrm{Prob}\,(y > z) < 1/3$. From Theorem 2.4, we deduce that $x\|y$ and $y\|z$ in P, and that $\mathrm{Prob}\,(x > y) + \mathrm{Prob}\,(y > z) \geqslant (5 - \sqrt{5})/5$, so $\mathrm{Prob}\,(x < y < z) \leqslant 1/\sqrt{5}$. We now make the two comparisons $x : y$ and $y : z$. We find one of: $x > y$, $y > z$, or $x < y < z$, each of which has probability at most $1/\sqrt{5}$. Thus, after two comparisons, we obtain a new poset P' with at most $L(P)/\sqrt{5}$ linear extensions. This leads to a contradiction as before. □

The reason we gain in the last part of the proof above is, loosely, that although the first comparison (say $x : y$) we make may not be "good enough", if we get the "bad" answer, then we know that the comparison $y : z$ will split the set of linear extensions very evenly. It seems almost certain that Theorem 8.1 can be improved by considering more and more elements that are close in average height; however, the analysis is bound to get more complicated. A proof of the 1/3–2/3 Conjecture would give $\phi_0 \leqslant 1/\log(3/2) \simeq 2.466$.

A lower bound on ϕ_0 is provided by finite segments of the infinite partial order Q defined in Section 7. The restriction Q_n of the partial order Q to $\{x_1, \ldots, x_n\}$ has F_n linear extensions, where F_n denotes the n'th Fibonacci number ($F_1 = 1$, $F_2 = 2$). So $\log L(Q_n)/n \to \log((1 + \sqrt{5})/2)$. It is easy to see that $S(Q_n) = n - 1$, since, in the worst case, all the $n - 1$ incomparable pairs of elements must be compared. Thus we have

$$\phi_0 \geqslant \lim_{n\to\infty} S(Q_n)/\log Q_n = 1/\log((1 + \sqrt{5})/2) \simeq 2.078.$$

We conjecture that this is in fact the correct value of ϕ_0, so that large finite segments of Q are indeed the "worst" posets to sort.

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